Unitarity and Cutkosky Rules in Nonlocal

非定域中的幺正性与库特科斯基规则

Quantum Field Theory

量子场论

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Abstract

摘要

In this section we discuss the unitarity of nonlocal quantum field theories. For simplicity, we focus our considerations on the case of a nonlocal scalar field, which encodes all the features of the nonlocality. We show that, for a specific class of nonlocal form factors, one can give a prescription for defining the scattering amplitudes, such that the theory is unitary. This prescription consists in defining the amplitudes in Euclidean signature, assuming that all the external and loop energies are purely imaginary. Then, physics is recovered by analytic continuation of such amplitudes to real and positive values of the external energies. We prove that the imaginary part of the analytically continued scattering amplitudes is given by the Cutkosky rules, which imply the unitarity of the theory. We conclude by showing that, if the nonlocal theory is defined directly in Minkowskian signature, assuming real external and loop energies, unitary is flawed.

在本节中我们讨论非局域量子场论的幺正性。为简化推导，我们将讨论聚焦于非局域标量场的情形，该模型已涵盖非局域性的所有核心特征。我们证明，对于特定类别的非局域形状因子，可以给出散射振幅的定义方案，使得该理论满足幺正性。该方案的内容是: 假设所有外能和圈能量均为纯虚数，在欧几里得号差下定义振幅；随后通过将这类振幅解析延拓到实正的外能取值，得到实际的物理结果。我们证明，解析延拓后的散射振幅的虚部满足卡特斯奇规则，该规则保证了理论的幺正性。最后我们指出，如果直接在闵氏号差下定义非局域理论、假设外能和圈能量均为实数，则理论不满足幺正性。

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Introduction

引言

Nonlocal quantum field theories have been extensively studied by Efimov [1-11] in the framework of the Bogolyubov axiomatic formulation of quantum field theory, to accomplish a finite realization of the Standard Model of particles or a confinement mechanism based on nonlocality [12]. Moreover, they are also relevant in the context of stochastic quantization [13, 14], where nonlocality is emergent. In this section, we are mainly interested in the role that nonlocality plays in the definition of a quantum field theory of gravitation, see [15] for a review of nonlocal gravity models, but our discussions do not depend on the specific field of application of nonlocal theories.

叶菲莫夫已在博戈留波夫量子场论公理化表述的框架内对非定域量子场论开展了广泛研究 [1-11]，以此实现粒子标准模型的有限实现，或基于非定域性的禁闭机制 [12]。此外，非定域量子场论还与随机量子化相关 [13,14]，非定域性正是在后者中涌现的。在本节中，我们主要研究非定域性在引力量子场论定义中发挥的作用，关于非定域引力模型的综述可参见 [15]，但我们的讨论不依赖于非定域理论的具体应用领域。

First renormalizable generalizations of the Einstein-Hilbert action have been realized by Stelle [24], Krasnikov [25], and Kuz'min [26] introducing fourth-order higher derivatives in the Lagrangian density of the gravitational field. However, in spite of being renormalizable, such models contain unavoidable ghosts that spoil unitarity. As unitarity is indispensable for the self-consistence of the theory, higher-derivative models were abandoned until it becomes clear that the occurrence of ghosts can be avoided introducing derivatives of infinite order in a proper manner (or, equivalently, considering a specific class of nonlocal interactions), and unitarity can be preserved defining the theory in Euclidean signature, according to a procedure that will be described in detail in the following sections.

斯塔勒 [24]、克拉斯尼科夫 [25] 和库兹明 [26] 首次实现了爱因斯坦-希尔伯特作用量的可重整推广，他们在引力场的拉格朗日密度中引入了四阶高阶导数。然而，尽管这类模型具备可重整性，却包含不可避免的鬼场，会破坏幺正性。由于幺正性是理论自洽的必要条件，高阶导数模型曾被弃用，直到人们发现: 以恰当方式引入无穷阶导数 (或等价地，考虑一类特殊的非定域相互作用) 就可以避免鬼场的出现，并且按照后续章节将详细描述的步骤，在欧几里得号差中定义理论即可保留幺正性。

For the sake of simplicity, in what follows, we limit our considerations to the paradigmatic example of nonlocal scalar field theory [1], which encodes all the features of nonlocal theories, at list for what concerns unitarity. Nonlocal scalar field theory is described by the following Lagrangian density:

为简便起见，下文我们将讨论限定在非定域标量场理论这一典型范例 [1]，该模型涵盖了非定域理论的所有特征，至少在幺正性相关方面是如此。非定域标量场理论由以下拉格朗日密度描述:

$$\mathcal{L}_\phi = -\frac{1}{2}\phi e^{H(-\sigma\Box)}\left(\Box + m^2\right)\phi - \frac{\lambda}{n!}\phi^n, \tag{1}$$

where $\Box = \eta^{\mu\upsilon}\partial_\mu\partial_\upsilon$ is the d' Alembert operator, $\eta^{\mu\upsilon}$ is the Minkowski metric tensor, and the last term in (1) represents an interaction with coupling constant λ . The term $\exp H(-\sigma\Box)$ in (1) is the nonlocal form factor, and $\sigma \in \mathbb{R}_0^+$ is a parameter with mass dimensions $[\sigma] = -2$ in natural units $\hbar = c = 1$, which fixes the scale of nonlocality as $\ell_\Lambda = \sqrt{\sigma}$. We will show later that, under some specific conditions on the nonlocal form factor (it must be entire with no zeroes for finite z), the model (1) is unitary and renormalizable. For reasons that will be clarified later, and without loss of generality, hereafter we set

其中 $\Box = \eta^{\mu\upsilon}\partial_\mu\partial_\upsilon$ 是达朗贝尔算符，$\eta^{\mu\upsilon}$ 是闵可夫斯基度量张量，式 (1) 的最后一项表示耦合常数为 λ 的相互作用。式 (1) 中的项 $\exp H(-\sigma\Box)$ 是非定域形状因子，$\sigma \in \mathbb{R}_0^+$ 是质量量纲为 $[\sigma] = -2$ 的参数 (在自然单位 $\hbar = c = 1$ 中)，将非定域的尺度固定为 $\ell_\Lambda = \sqrt{\sigma}$。我们后续会证明，在非定域形状因子满足特定条件 (它必须是整函数，且在有限 z 处无零点) 的情况下，模型 (1) 是幺正且可重整的。出于后续会阐明的原因，在不失一般性的前提下，我们在此设

$$H\left(\sigma m^2\right) = 0. \tag{2}$$

In fact, this condition can always be met by multiplying the Lagrangian density (1) by a positive number $e^{-H(\sigma m^2)}$ and redefining $H(z)$ and the coupling constant as $\widetilde{H}(z) \equiv H(z) - H\left(\sigma m^2\right)$ and $\tilde{\lambda} \equiv \lambda e^{-H(\sigma m^2)}$.

实际上，这个条件总能通过将拉格朗日密度 (1) 乘以一个正数 $e^{-H(\sigma m^2)}$，再将 $H(z)$ 和耦合常数重新定义为 $\widetilde{H}(z) \equiv H(z) - H\left(\sigma m^2\right)$ 和 $\tilde{\lambda} \equiv \lambda e^{-H(\sigma m^2)}$ 来满足。

Before proceeding further, let us underline some features of (1). First, we emphasize that the nonlocal form factor introduces higher derivatives, actually derivatives of arbitrarily high order in physically interesting cases. Moreover, since $\eta_{\mu\nu}$ is the Minkowski tensor,(1) represents a scalar field in flat space-time. This is enough for our purposes as we are only interested in checking the compatibility between nonlocality and unitarity, and we do not aim to introduce gravitational interactions at this stage. Furthermore, derivatives of any order enter the Lagrangian through the d'Alembertian operator $\Box$, and this fact implies that the theory is Lorentz invariant.

在继续推导之前，我们先强调一下式 (1) 的几个特点。首先我们需要指出，非定域形状因子会引入高阶导数，在有物理意义的情形中实际上是任意高阶的导数。此外，由于 $\eta_{\mu\nu}$ 是闵可夫斯基张量，式 (1) 描述的是平坦时空中的标量场。这足以满足我们的研究目的，因为我们仅需要检验非定域性和幺正性的相容性，现阶段并不旨在引入引力相互作用。另外，任意阶导数都通过达朗贝尔算符 $\Box$ 进入拉格朗日量，这说明该理论满足洛伦兹不变性。

Let us now define a field φ as

现在我们定义场 φ 为

$$\varphi = e^{\frac{1}{2}H(-\sigma\Box)}\phi. \tag{3}$$

It is easy to show that (3) can be inverted if the function $e^{H(z)}$ has no zeros. In fact, Fourier transforming (3), one has the relation

不难证明，若函数 $e^{H(z)}$ 没有零点，式 (3) 就可以求逆。对式 (3) 做傅里叶变换后可得关系

$$\widetilde{\varphi}\left(k^2\right) = e^{\frac{1}{2}H(\sigma k^2)}\widetilde{\phi}\left(k^2\right) \tag{4}$$

between the Fourier transforms $\widetilde{\varphi}$ and $\widetilde{\phi}$ of the two fields φ and ϕ , respectively, and this relation is invertible provided that $e^{H(\sigma k^2)} \neq 0 \forall k^2$. We will see later that this condition, namely that the nonlocal form factor has no zeros for finite values of its argument, has a deep connection with unitarity. Indeed, under this assumption, (3) is invertible, and the Lagrangian (1) can be recast in terms of the field φ as

分别在两个场 φ 和 ϕ 的傅里叶变换 $\widetilde{\varphi}$ 和 $\widetilde{\phi}$ 之间，当满足 $e^{H(\sigma k^2)} \neq 0 \forall k^2$ 时该关系可逆。我们稍后会看到，这个条件即非局部形状因子在其自变量取有限值时不存在零点，与幺正性有深刻关联。事实上，在该假设下，式 (3) 可逆，拉格朗日量 (1) 可以用场 φ 改写为

$$\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{n!}\left(e^{-\frac{1}{2}H(-\sigma\Box)\varphi}\right)^n, \tag{5}$$

so that the models (1) and (5) are the equivalent. We note that the nonlocal model (5) has the same kinetic term as a local scalar theory, while nonlocality is restricted to the interaction term. This implies that a free nonlocal scalar field is equivalent to a free local scalar field, and nonlocality cannot exist without interactions. Moreover, one can express the field ϕ as

因此模型 (1) 和模型 (5) 是等价的。我们注意到非局部模型 (5) 的动能项与局域标量理论一致，非局域性仅存在于相互作用项中。这说明自由非局部标量场等价于自由局域标量场，非局域性不能脱离相互作用存在。此外，场 ϕ 可以表示为

$$\phi(x) = e^{-\frac{1}{2}H(-\sigma\Box)}\varphi = \int \frac{d^4y}{(2\pi)^4} K(x-y)\varphi(y), \tag{6}$$

that is, the convolution between the field φ and the function

即场 φ 与该函数的卷积

$$K(x) = \int \frac{d^4k}{(2\pi)^4} e^{-\frac{1}{2}H(-\sigma k^2)} e^{ikx}. \tag{7}$$

Therefore, the interaction potential is rewritten as

因此，相互作用势可改写为

$$V_{\text{Int}} = \frac{\lambda}{n!}\left(\int \frac{d^4y}{(2\pi)^4} K(x-y)\,\varphi(y)\,\varphi\right)^n, \tag{8}$$

and it depends on the values that the field φ takes at any point of the space-time, so that interactions are nonlocal. This is why the class of models (1) and (5) are referred as nonlocal quantum field theories.

它依赖于场 φ 在时空任意点的取值，因此相互作用是非局部的。这就是模型 (1) 和 (5) 这类理论被称为非局部量子场论的原因。

In what follows, we discuss the conditions under which the nonlocal scalar fields (1) and (5) are unitary. We show that the nonlocal form factor $e^{H(z)}$ must be an entire function (analytic with no poles at finite z), which has no zeroes at finite complex values of z . Moreover, the scattering amplitudes must be defined in Euclidean signature, assuming that all the external and loop energies are purely imaginary, and then continued analytically to real values of the external energies. We mention that the unitarity of nonlocal scalar fields has been proven long time ago in the context of the axiomatic formulation of quantum field theory, see [10]. However, here we adopt a different perspective outlined in a series of papers [11, 29-33], which is based on the generalization of Cutkosky rules to nonlocal fields. In section ”Unitarity” we review the concept of unitarity of a quantum field theory. There we show that unitarity is guaranteed if the complex parts of all the scattering amplitudes satisfy the condition (14). Then, in section ”Feynman Rules and the Nonlocal Propagator” we give the Feynman rules for the nonlocal theories (1) and (5). In section ”Complex Amplitudes and Cutkosky Rules” we analyze the properties of the scattering amplitudes in nonlocal scalar theories. First, we show how to define them in Euclidean signature and recover physical scattering amplitudes by means of their analytic continuation. In doing so, with illustrative purposes, in section ”One-Loop Diagram” we consider a simple example of a one-loop Feynman diagram, for which the analytic continuation of the scattering amplitude is evaluated explicitly. Also, it is shown that the imaginary part of this amplitude is the same as that of the local scalar theory, and it is given by the Cutkosky rules. In section ”Cutkosky Rules, Normal and Anomalous Thresholds, and Unitarity”, the Cutkosky rules are defined for generic nonlocal scattering amplitudes. Moreover, normal and anomalous thresholds are defined, and their role for unitarity is discussed. It is argued that, if only normal thresholds contribute to the imaginary part of the scattering amplitudes, and this is given by the application of the Cutkosky rules, the theory is unitary. Again, with pedagogical purposes, in section ”One-Loop Amplitude with an Anomalous Threshold”, a one-loop diagram with an anomalous threshold is considered. The imaginary part of the corresponding nonlocal amplitude is computed explicitly, showing that it is not affected by the anomalous threshold. These considerations are generalized to arbitrary Feynman diagrams in section ”Generic Amplitudes”, where we show that the Cutkosky rules are valid for generic nonlocal scattering amplitudes, and anomalous thresholds do not contribute to their imaginary part, and indeed the nonlocal theories (1) and (5) are unitary. Finally, in section "Non-unitarity of the Minkowskian Theory" we show that if the nonlocal theory is defined in Minkowskian signature, so that the scattering amplitudes are defined for real external and loop energies, the Cutkosky rules are no longer valid, and the unitarity of the theory is spoiled.

在下文中，我们将讨论非局域标量场 (1) 和 (5) 满足幺正性的条件。我们证明，非局域形状因子 $e^{H(z)}$ 必须是整函数 (在有限 z 处解析且无极点)，且在有限复值 z 处无零点。此外，散射振幅必须在欧几里得号差下定义: 假设所有外能量和圈能量均为纯虚数，之后再解析延拓到外能量的实数值。我们指出，非局域标量场的幺正性早在量子场论公理化表述的框架下就已得到证明，参见文献 [10]。但本文我们采用一系列文献 [11, 29-33] 中提出的不同视角，该视角基于将卡茨基规则推广到非局域场。在“幺正性”一节中，我们回顾量子场论幺正性的概念。我们将在该节说明，若所有散射振幅的复部满足条件 (14)，则幺正性得到保证。随后在“费曼规则与非局域传播子”一节中，我们给出非局域理论 (1) 和 (5) 的费曼规则。在“复振幅与卡茨基规则”一节中，我们分析非局域标量理论中散射振幅的性质。我们首先说明如何在欧几里得号差下定义这些振幅，并通过解析延拓得到物理散射振幅。为举例说明，我们在“单圈图”一节中研究了一个简单的单圈费曼图例子，并对其散射振幅的解析延拓做了显式计算。我们还证明，该振幅的虚部与局域标量理论的虚部一致，且由卡茨基规则给出。在“卡茨基规则、正规阈、反常阈与幺正性”一节中，我们为一般非局域散射振幅定义了卡茨基规则，还定义了正规阈与反常阈，并讨论了它们对幺正性的作用。我们提出: 若只有正规阈对散射振幅的虚部有贡献，且虚部由应用卡茨基规则得到，则该理论是幺正的。同样出于教学目的，我们在“带反常阈的单圈振幅”一节中研究了一个带反常阈的单圈图。我们对相应非局域振幅的虚部做了显式计算，结果表明虚部不受反常阈影响。这些考虑在“一般振幅”一节中被推广到任意费曼图: 我们证明卡茨基规则适用于一般非局域散射振幅，反常阈不对其虚部产生贡献，因此非局域理论 (1) 和 (5) 确实满足幺正性。最后，我们在“闵氏理论的非幺正性”一节中证明: 若非局域理论定义在闵可夫斯基号差下，即散射振幅对实外能量和实圈能量定义，则卡茨基规则不再成立，理论的幺正性也会被破坏。

Unitarity

幺正性

The validation of a quantum field theory as a model for elementary particles is made through the calculation of cross sections and their comparison with data from scattering experiments, see [27,28] for an exhaustive review of local quantum field theory. Cross sections, in turn, are calculated via transition amplitudes between an initial incoming state $|i\rangle$ and an outgoing final state $|f\rangle$ that are defined asymptotically in the far past and future, respectively. Indeed, for a given scattering process $i \to f$, one has

量子场论作为基本粒子模型的有效性通过计算散射截面并与散射实验数据比对验证，关于局域量子场论的详尽综述参见文献 [27,28]。而散射截面是通过分别在遥远过去和遥远未来渐近定义的初入射态 $|i\rangle$ and an outgoing final state $|f\rangle$ 之间的跃迁振幅计算得到的。对于给定的散射过程 $i \to f$，有

$$\langle f \mid i \rangle = \lim_{t\to\infty} \langle b \,|e^{-2itH}|\, a\rangle \equiv \langle b\,|S|\,a\rangle = S_{ba}, \tag{9}$$

where we have defined the initial and final states by means of time translation of eigenstates $|a\rangle$ and $|b\rangle$

of the Hamiltonian H corresponding to a certain number of particles with definite momentum in the far past and future respectively, that is,

其中我们通过哈密顿量 H 的本征态 $|a\rangle$ and $|\,b\rangle$ 的时间平移定义了初态和末态，这些本征态分别对应遥远过去和遥远未来中具有确定动量的一定数目粒子，即

(10)

$$\Big|i\rangle = \lim_{t\to+\infty} e^{-itH}\Big|\,a\rangle,$$

$$\Big|f\rangle = \lim_{t\to-\infty} e^{-itH}\Big|\,b\rangle = \lim_{t\to+\infty} e^{itH}|b\rangle.$$

The operator S defined in (9) is called S matrix. As it is given by a sequence of unitary operators, the S matrix is unitary, i.e., it satisfies the unitarity condition

式 (9) 中定义的算符 S 被称为 S 矩阵。由于它由一系列幺正算符构成，因此 S 矩阵是幺正的，即它满足幺正条件

$$S^\dagger S = 1 \tag{11}$$

The unitarity condition (11) is then expressed in terms of the T matrix as

因此幺正条件 (11) 可以用 T 矩阵表示为

$$T - T^\dagger = iT^\dagger T \tag{12}$$

where the T matrix is defined by the relation $S \equiv 1 + iT$, and it represents the part of S that gives rise to scattering processes. Taking the expectation value of (12) between $|a\rangle$ and $\langle b|$ as in (11) for a given process $i \to f$, one has

其中 T 矩阵由关系 $S \equiv 1 + iT$ 定义，它代表 S 中产生散射过程的部分。对给定过程 $i \to f$ ，取 (11) 中 $|a\rangle$ and $\langle b|$ 之间 (12) 式的期望值，可得

$$T_{ba} - T^*_{ab} = i\left\langle b\left|T^\dagger T\right|a\right\rangle = i\sum_n \int \prod_{i=1}^n \frac{d^3q_i}{2E_{q_i}(2\pi)^3} T^*_{nb}T_{na}, \tag{13}$$

where $T_{na} = \left\langle n\left|T^\dagger T\right|a\right\rangle, T_{nb} = \left\langle n\left|T^\dagger T\right|b\right\rangle, |n\rangle \equiv \left|q_1, \dots, q_n\right\rangle$ is a complete set of | intermediate n-particle states of definite momenta, and $E_{q_i} \equiv \sqrt{\vec{q}_i + m^2}$.

其中 $T_{na} = \left\langle n\left|T^\dagger T\right|a\right\rangle, T_{nb} = \left\langle n\left|T^\dagger T\right|b\right\rangle, |n\rangle \equiv \left|q_1, \dots, q_n\right\rangle$ is a complete set of | 是具有确定动量的中间 n 粒子态，且 $E_{q_i} \equiv \sqrt{\vec{q}_i + m^2}$ 。

Rearranging the matrix elements of the T matrix in terms of those of the invariant scattering amplitude $\mathcal{M}$ as $T_{ab} = (2\pi)^4\mathcal{M}_{ab}\delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right)$, where $\mathcal{M}_{ba} = \langle b\,|\mathcal{M}|\,a\rangle$, and p_i and p_f are the external 4-momenta of particles in the initial and final states, (13) is then recast as

将 T 矩阵的矩阵元重新整理为不变散射振幅 $\mathcal{M}$ 的矩阵元，形如 $T_{ab} = (2\pi)^4 \mathcal{M}_{ab} \delta^{(4)}\left(\sum_i p_i - \sum_f p_f\right)$，其中 $\mathcal{M}_{ba} = \langle b | \mathcal{M} | a \rangle$， p_i 和 p_f 分别是初态和末态中粒子的外部四动量，整理后 (13) 可改写为

$$\mathcal{M}_{ba} - \mathcal{M}_{ab}^* = i \sum_n \int \prod_{i=1}^{n} \frac{d^3 q_i}{2E_{q_i} (2\pi)^3} \mathcal{M}_{nb}^* \mathcal{M}_{na} (2\pi)^4 \delta^{(4)}(p_c - p_a), \tag{14}$$

which is our final version of the unitarity condition. The interpretation of (14) is that the imaginary part of the scattering amplitude of the process $a \to b$ is given by a sum of products of the amplitudes of the processes $a \to n$, where n is an intermediate state, and the complex conjugate of the amplitude of the corresponding processes $b \to n$. That means that the 1. h. s. of (14) is nonzero only when the processes $a \to n$ and $b \to n$ are kinematically permitted. Indeed, the imaginary part of $\mathcal{M}_{ab}$ is related to the opening of a channel of production of real intermediate states n , which typically happens above some threshold energy. This observation is strictly related to the Cutkosky rules that will be introduced in the next section ”Complex Amplitudes and Cutkosky Rules”.

这就是我们最终形式的幺正条件。式 (14) 的含义是: 过程 $a \to b$ 散射振幅的虚部等于过程 $a \to n$ (其中 n 为中间态) 振幅乘积与对应过程 $b \to n$ 振幅复共轭的乘积之和。这意味着仅当过程 $a \to n$ 和 $b \to n$ 满足运动学允许条件时，式 (14) 的左端才非零。事实上，$\mathcal{M}_{ab}$ 的虚部与实中间态 n 产生通道的开启相关，这通常发生在能量高于某一阈值时。该结论与下一节“复振幅与 Cutkosky 规则”将要介绍的 Cutkosky 规则密切相关。

Our task for the next sections consists in generalizing the Feynman rules to the nonlocal models (1) and (5) in Euclidean signature, performing the analytic continuation to real and positive external energies, and showing that the resulting complex amplitudes satisfy the unitary condition (14). This will be accomplished generalizing the so-called Cutkosky rules to nonlocal quantum fields.

我们后续章节的任务是: 将费曼规则推广到欧几里得记号下的非定域模型 (1) 和 (5)，对实正外部能量做解析延拓，并证明得到的复振幅满足幺正条件 (14)。我们将通过把著名的 Cutkosky 规则推广到非定域量子场来完成这一目标。

Feynman Rules and the Nonlocal Propagator

费曼规则与非局部传播子

In this section, we discuss the Feynman rules for the nonlocal scalar fields (1) and (5). Let us start considering the Lagrangian density (1). As we will see, our conclusions will be still valid for the equivalent model (5). We define the theory in Euclidean signature, so that all the time components of the 4-momenta are assumed to be purely imaginary. Moreover, in our notation,

本节我们讨论 (1) 和 (5) 中非局部标量场的费曼规则。我们先从拉格朗日密度 (1) 开始分析，我们的结论对等价模型 (5) 同样成立，正如我们之后会看到的。我们在欧几里得号差下定义该理论，因此假设所有四动量的时间分量都是纯虚数。此外，在我们的记号中，

$$p^2 = \left(p^0\right)^2 - \vec{p}^2. \tag{15}$$

The Feynman rules for this theory can be derived by means of functional methods from Green functions defined by path integrals. They are the same as in the local theory, except from the fact that the scalar propagator in momentum representation now reads

> 该理论的费曼规则可通过泛函方法，由路径积分定义的格林函数推导得到。除了动量空间的标量传播子现取下述形式外，其余规则都与局部理论一致:

$$D(q) = \frac{ie^{-H(\sigma p^2)}}{p^2 - m^2 + i\varepsilon}, \tag{16}$$

with a clear dependence on the nonlocal form factor. For our convenience, we have introduced an $i\varepsilon$ term in the propagator, with $\varepsilon > 0$ infinitesimal, which is irrelevant in Euclidean signature, since $p^2 - m^2 < 0$ for purely imaginary p^0 and $m \neq 0$. However, this term will be important to recover physical amplitudes, when we will send all the external energies to real and positive values in the limit $\varepsilon \to 0$. We also note that the local theory is recovered by setting $H(z) \equiv 0$.

> 传播子明显依赖于非局部形状因子。为方便起见，我们在传播子中引入了 $i\varepsilon$ 项，其中 $\varepsilon > 0$ 是无穷小量，该量在欧几里得号差下无关紧要，因为对纯虚数 p^0 和 $m \neq 0$ 有 $p^2 - m^2 < 0$ 。不过当我们在 $\varepsilon \to 0$ 极限下将所有外能推至实的正值以恢复物理振幅时，该项会起到重要作用。我们还注意到，令 $H(z) \equiv 0$ 即可得到局部理论。

From (16) we can infer two features of nonlocal theories. First, if one chooses $H(z)$ in such a way that

> 从 (16) 式我们可以推导出非局部理论的两个性质。首先，如果选择 $H(z)$ 使得

$$\lim_{z \to -\infty} e^{-H(z)} = 0 \tag{17}$$

in the ultraviolet limit $-p^2 \to +\infty$ the propagator goes to zero faster than in the local case, so that the convergence of complex amplitudes is improved. This is at the basis of the renormalizability of nonlocal gravitational models, see [15] and [16-23]. To explain this fact, we can use usual power counting techniques. Consider a generic Feynman diagram with L loops, P internal lines, and V vertices. We assume that the nonlocal form factor diverges as a power of its argument in the ultraviolet limit, so that

> 在紫外极限 $-p^2 \to +\infty$ 下，传播子比局部情况更快趋于零，因此复振幅的收敛性得到改善。这是非局部引力模型可重整化的基础，参见文献 [15] 和 [16-23]。我们可以用常规幂次计数技术解释这一点。考虑一个拥有 L 个圈、P 条内线和 V 个顶点的一般费曼图。我们假设非局部形状因子在紫外极限下按自变量的幂次发散，即

$$\lim_{z \to -\infty} z^{\gamma} e^{-H(z)} = C, \tag{18}$$

for some $\gamma > 0$, where $C \neq 0$ is a constant. Thus, the propagator (16) scales as $p^{-2(\gamma+1)}$ at large p^2 , and the superficial degree of divergence δ of the diagram, see [27, 28] for review, is given by

对某个 $\gamma > 0$ 成立，其中 $C \neq 0$ 是常数。因此，传播子 (16) 在大 p^2 下标度为 $p^{-2(\gamma+1)}$，该图的表观发散度 δ (综述参见 [27, 28]) 由下式给出:

$$\delta = 4L - 2(\gamma + 1) P. \tag{19}$$

At that point, we can use the expression $L = P - V + 1$ that gives the number of loops in terms of P and the number of vertices V , and the topological relation $nV = N + 2P$, where N is the number of external lines of the diagram, to obtain

至此，我们可以利用用 P 和顶点数 V 表示圈数的表达式 $L = P - V + 1$ ，以及拓扑关系 $nV = N + 2P$ (其中 N 是图的外线数)，得到:

$$\delta = 4 - \frac{4}{n}N - 2\left(\gamma - 1 + \frac{4}{n}\right) P. \tag{20}$$

Thus, if $\gamma > 1 - 4/n$, the theory is renormalizable or even finite. In that way, an interaction term with $n > 4$, that is nonrenormalizable in the local theory, can be rendered renormalizable by introducing the nonlocality.

因此，如果 $\gamma > 1 - 4/n$ ，理论就是可重整化的，甚至是有限的。通过这种方式，在局部理论中不可重整化、带有 $n > 4$ 的相互作用项，可以通过引入非局域性变为可重整化。

The second feature is that if the nonlocal form factor is such that $e^{H(\sigma p^2)} \neq 0$ at any finite p^2 , the only poles of the nonlocal propagator (16) are those of the local theory at $p^2 = m^2$. Since physical particles correspond to the poles of the propagator, which means that the nonlocal theory has the same degrees of freedom on the local scalar field, indeed it cannot contain ghosts or, in other words, it must be unitary. On the contrary, in Pauli-Villars regularization scheme [27], corresponding to the choice $e^{H(-\sigma\Box)} = \left(\Box + \Lambda^2\right) / \left(\Lambda^2 - m^2\right)$, one has

第二个性质是，若非局部形状因子满足对任意有限 p^2 都有 $e^{H(\sigma p^2)} \neq 0$ ，则非局部传播子 (16) 仅拥有和局部理论相同的极点，位于 $p^2 = m^2$ 处。由于物理粒子对应传播子的极点，这意味着非局部理论在局部标量场上拥有相同的自由度，它确实不包含鬼场，换句话说，该理论一定是幺正的。相反，在泡利-维尔斯正则化方案 [27](对应选择 $e^{H(-\sigma\Box)} = \left(\Box + \Lambda^2\right) / \left(\Lambda^2 - m^2\right)$) 中，我们有:

$$D(q) = \frac{i\left(m^2 - \Lambda^2\right)}{(p^2 - m^2 + i\varepsilon)(p^2 - \Lambda^2 + i\varepsilon)} = \frac{i}{(p^2 - m^2 + i\varepsilon)} - \frac{i}{(p^2 - \Lambda^2 + i\varepsilon)},$$

(21)

so that the scalar field propagator has an extra pole at $p^2 = \Lambda^2$, corresponding to a propagating massive ghost, whose propagator (with the typical ghostly wrong sign) is given by the last term in (21). As we will see in the following, this naive argument is right, as far as the theory is defined in Euclidean signature and then continued to real and positive external energies. It fails instead, if the theory is defined from the beginning in Minkowskian signature, assuming real external and loop energies.

因此标量场传播子在 $p^2 = \Lambda^2$ 处存在一个额外极点，对应一个可传播的有质量鬼场，其传播子 (带有典型的鬼场错误符号) 由式 (21) 的最后一项给出。下文我们将会看到，只要该理论是在欧几里得号差下定义再解析延拓到实正的外能，这个直观论证就是正确的。但如果理论从一开始就在闵可夫斯基号差下定义，且假设外能和圈动量都是实的，该论证就不成立。

Hereafter we assume that the nonlocal form factor $e^{H(z)}$ is an entire function (an analytic function with no poles at finite z), which has no zeroes for finite z . We ask that $e^{H(z)}$ has no zeros in the finite z plane to avoid ghosts, and we impose that it is entire, since the propagator cannot be zero at finite momenta for propagating particles.

下文我们假设非局部形状因子 $e^{H(z)}$ 是整函数 (即在有限 z 处没有极点的解析函数)，且在有限 z 处没有零点。我们要求 $e^{H(z)}$ 在有限 z 平面没有零点以避免鬼场，同时要求它是整函数，因为传播子不能对可传播粒子在有限动量处为零。

We note that, under these conditions, the field redefinition (3) is invertible and the two formulations (1) and (5) of the nonlocal scalar field are equivalent. This equivalence can be pushed forward showing that, although formally different, the Feynman rules for the theories (1) and (5) give identical scattering amplitudes for a given diagram. In fact, for the nonlocal scalar field (5), the φ propagator is the same as that of the corresponding local theory, while the Feynman rules prescribe to associate to each vertex a term

我们注意到，在这些条件下，场重定义 (3) 是可逆的，非局部标量场的两种表述 (1) 和 (5) 是等价的。我们可以进一步推进这一等价性，证明尽管形式不同，理论 (1) 和 (5) 的费曼规则对给定图给出完全相同的散射振幅。实际上，对于非局部标量场 (5)，φ 传播子与对应局域理论的传播子完全一致，而费曼规则要求给每个顶点关联一个项

$$\lambda \mathcal{V}\left(p_1, p_2, \ldots, p_n\right) = \lambda \prod_{i=1}^{n} e^{-\frac{1}{2} H\left(\sigma(p_i)^2\right)}, \tag{22}$$

where $p_1^{(j)}, \ldots p_n^{(j)}$ are the n 4-momenta at the j vertex. Indeed, it is easy to realize (e.g., by direct inspection of simple diagrams) that the two sets of Feynman rules for the models (1) and (5), in which the nonlocal form factor is attached to the propagator and to the vertex, respectively, give the same scattering amplitudes. This is shown explicitly in (23) and (26) in the case of the one-loop amplitude studies in section ”One-Loop Diagram”.

其中 $p_1^{(j)}, \ldots p_n^{(j)}$ 是 j 顶点处的 n 四动量。不难发现 (例如通过对简单图直接检验)，模型 (1) 和 (5) 分别将非局部形状因子绑定在传播子和顶点上，这两套费曼规则给出的散射振幅是相同的。在“单圈图”一节研究单圈振幅时，我们会在式 (23) 和 (26) 中明确给出这一结论的证明。

Complex Amplitudes and Cutkosky Rules

复振幅与卡茨基规则

In this section we show how to compute the complex amplitudes for the nonlocal scalar field theories (1) and (5) in the Euclidean space, that is, for purely imaginary external and loop energies, and perform their analytic continuation to real and positive external energies. Moreover, we give a prescription, commonly referred as Cutkosky rules in the case of local theories, for evaluating the imaginary part of such complex amplitudes. The Cutkosky rules are then used to prove the unitarity of the nonlocal scalar field theory, showing that the nonlocal scattering amplitudes satisfy the condition (14).

> 在本节中，我们将说明如何计算欧几里得空间中非局域标量场论 (方程 1 和方程 5) 的复振幅——即针对全纯虚外能与圈能的情况——并对其进行解析延拓得到实正外能。此外，我们给出了计算这类复振幅虚部的一套规则，在局域场论中这套规则通常被称为卡茨基规则。我们随后使用卡茨基规则证明了非局域标量场论的幺正性，表明非局域散射振幅满足条件 (14)。

In Euclidean signature, the scattering amplitudes are calculated integrating the loop energies k_i^0 along the imaginary axis $\mathcal{I}$ of the complex k_i^0 plane, while the energies of the external particles are taken to be purely imaginary. In this situation, the poles of the propagators are far from the integration contour. Moreover, if the theory is finite, the loop integrals are supposed to converge due to the ultraviolet behavior of the nonlocal form factor, see (17)-(18). Indeed, the complex amplitudes are not singular for purely imaginary external energies. If the theory is renormalizable or super-renormalizable, one should apply the same arguments to renormalized diagrams.

> 在欧几里得号差下，我们计算散射振幅时，是沿复 k_i^0 平面的虚轴 $\mathcal{I}$ 对圈能量 k_i^0 积分，同时将外粒子能量取为纯虚数。在这种设定下，传播子极点远离积分围道。此外，如果理论有限，由于非局域形状因子的紫外行为，圈积分应当收敛，参见 (17)-(18)。事实上，对于纯虚外能，复振幅没有奇点。如果理论是可重整或超可重整的，上述论证同样适用于重整化后的图。

The analytic continuation of these amplitudes is obtained by moving the energies of the external particles to real and positive values. In doing so, some of the poles of the propagators move through the complex k_i^0 plane, possibly crossing the imaginary axis $\mathcal{I}$. Therefore, the analytic continuation is performed deforming the integration contours $\mathcal{I}$ in the loop energies to new contours $\mathcal{C}_i$ of the complex k_i^0 plane, in order to avoid these moving poles. It typically happens that, in the limit $\varepsilon \to 0$ and for some real values of the external energies, some of the poles of the propagators coincide. If it also happens that the deformed integration contour $\mathcal{C}$ is constrained to pass between two coinciding poles, it cannot be deformed further, and we say that $\mathcal{C}$ is pinched by these poles. In this situation, the complex amplitude develops a branch-cut singularity in the external energies. When the external energies cross this singularity, the imaginary part of the complex amplitude has a jump. As we have discussed before,(14) implies that when then the amplitudes $\mathcal{M}_{na}$ and $\mathcal{M}_{nb}$ for the intermediate processes $a \to n$ and $a \to n$ become nonzero, which means that the intermediate state n can really be produced above some energy threshold, the imaginary part of the complex amplitude acquires a new term. This means that each branch-cut singularity corresponds to the threshold of production of new intermediate n states.

> 这些振幅的解析延拓通过将外粒子能量移动到实正值得到。在此过程中，部分传播子极点在复 k_i^0 平面移动，有可能穿过虚轴 $\mathcal{I}$ 。因此，我们做解析延拓时会将圈能量的积分围道 $\mathcal{I}$ 变形为复 k_i^0 平面上的新围道 $\mathcal{C}_i$ ，以避开这些移动的极点。通常会出现这种情况: 在极限 $\varepsilon \to 0$ 下，当外粒子能量取某些实数值时，部分传播子极点重合。若此时变形后的积分围道 $\mathcal{C}$ 必须从两个重合极点之间穿过，就无法继续变形，我们称 $\mathcal{C}$ 被这些极点夹住了。这种情况下，复振幅会在外能空间中出现割线奇点。当外能穿过该奇点时，复振幅的虚部会发生跳变。正如我们之前讨论的，(14) 表明，此时中间过程 $a \to n$ 对应的振幅 $\mathcal{M}_{na}$ 和 $\mathcal{M}_{nb}$ 非零，这意味着中间态 n 确实可以在高于能量阈值时产生，复振幅的虚部会因此新增一项，即每个割线奇点都对应产生新中间 n 态的能量阈值。

In the next sections we will show how to evaluate the imaginary part of $\mathcal{M}_{ba}$ at a branch-cut singularity using the residue theorem. This will lead us to the formulation of the Cutkosky rules for nonlocal scalar fields and then to the proof of their unitarity. In order to clarify the concepts discussed in this section, we will first consider the case of simple one-loop diagrams, and then we will analyze generic scattering amplitudes.

在后文中我们将说明如何利用留数定理计算割线奇点处 $\mathcal{M}_{ba}$ 的虚部。这将引导我们给出非局域标量场的卡茨基规则，进而证明其幺正性。为了阐明本节讨论的概念，我们将先分析简单单圈图的情况，再分析一般散射振幅。

One-Loop Diagram

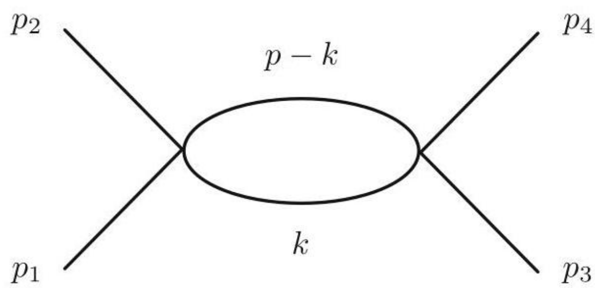
单圈图

With illustrative purposes, we start considering a simple one-loop diagram, so we can show explicitly how to define the corresponding scattering amplitude in Euclidean signature. Moreover, we use this example to clarify the procedure of analytic continuation to real external energies. We will show that this diagram has a branch-cut singularity, and its imaginary part at this singularity is given by the Cutkosky rules.

出于说明目的，我们首先研究一个简单的单圈图，借此明确展示如何在欧几里得号差下定义对应的散射振幅。此外，我们通过该示例阐明解析延拓到实外部能量的步骤。我们将证明该图存在一个分支切割奇点，其在该奇点处的虚部由 Cutkosky 规则给出。

Fig. 1 One-loop contribution to the four-particle scattering amplitude in $\lambda\phi^4/4\,!$ theory

图 1 $\lambda\phi^4/4\,!$ 理论中四粒子散射振幅的单圈贡献



Let us consider the Feynman diagram depicted in Fig. 1 that represents the one-loop contribution to the scattering process $\phi+\phi\to\phi+\phi$ in a nonlocal scalar field theory with quartic interaction, that is (1) or (5) with $n=2$. The corresponding scattering amplitude is calculated by means of the Feynman rules for the nonlocal scalar field discussed in section "Feynman Rules and the Nonlocal Propagator". Using these rules for the theory (5), that is, attaching the nonlocal form factor to the vertex, one has (hereafter, we omit the indices ab in $\mathcal{M}_{ab}$)

我们考虑图 1 所示的费曼图，它代表带四次相互作用的非局域标量场论中散射过程 $\phi+\phi\to\phi+\phi$ 的单圈贡献，即 $n=2$ 对应的式 (1) 或式 (5)。我们借助“费曼规则与非局域传播子”一节讨论的非局域标量场的费曼规则计算对应散射振幅。将这些规则应用于理论 (5)——也就是将非局域形状因子附在顶点上，可得 (下文中我们省略 $\mathcal{M}_{ab}$ 中 ab 的指标)

$$\mathcal{M}(p_h,\varepsilon)=-\frac{\lambda^2}{2}\int_{(\mathcal{I}\times\mathbb{R}^3)}\frac{id^4k}{(2\pi)^4}\,\frac{\mathcal{V}(p_1,p_2,k,p-k)}{k^2-m^2+i\varepsilon}\,\frac{\mathcal{V}(p-k,k,p_3,p_4)}{(k-p)^2-m^2+i\varepsilon},$$

(23)

where p_h represents the external 4-momenta p_1,p_2,p_3,p_4 , and $p=p_1+p_2=\ p_3+p_4$. The vertex function $\mathcal{V}$ is defined according to (22), so that

其中 p_h 表示外部四动量 p_1,p_2,p_3,p_4 和 $p=p_1+p_2=\ p_3+p_4$。顶点函数 $\mathcal{V}$ 根据式 (22) 定义，因此有

$$\mathcal{V}(v_1,v_2,v_3,v_4)=e^{-\frac{1}{2}\sum_{i=1}^{4}H\left(\sigma(v_i)^2\right)}. \tag{24}$$

Since we are in Euclidean signature, the loop energy k^0 , that is, the time component of the loop 4-momenta k , is purely imaginary, and the k^0 integration in (23) is performed along the imaginary axis $\mathcal{I}$ of the complex k^0 plane. Also, the external energies p_1^0,p_2^0,p_3^0 , and p_4^0 are purely imaginary. They will be moved to real values when we will perform the analytic continuation of (23).

由于我们处于欧几里得号差下，圈能量 k^0 (即圈四动量 k 的时间分量) 是纯虚数，式 (23) 中对 k^0 的积分沿复 k^0 平面的虚轴 $\mathcal{I}$ 进行。此外，外部能量 p_1^0,p_2^0,p_3^0 和 p_4^0 也都是纯虚数。当我们对式 (23) 做解析延拓时，它们会被移动到实数值。

We note that if one uses the Feynman rules for the model (1), one obtains the same expression (23). In fact, using (2), and the fact that external momenta p_1,p_2 , p_3 , and p_4 are on-shell, one has

我们注意到，若使用模型 (1) 的费曼规则，会得到和式 (23) 完全相同的表达式。实际上，利用式 (2) 以及外部动量 p_1,p_2 、 p_3 和 p_4 都在壳的性质，可得

$$\mathcal{V}(p_1,p_2,k,p-k)\,\mathcal{V}(p-k,k,p_3,p_4)=e^{-H\left(\sigma(k)^2\right)}e^{-H\left(\sigma(p-k)^2\right)}. \tag{25}$$

Replacing this expression in (23), one obtains the same complex amplitude given by the propagator (16) and local vertices, namely

将该表达式代入式 (23)，就得到由传播子 (16) 和局域顶点给出的同一个复振幅，即

$$\mathcal{M}(p_h,\varepsilon)=-\frac{\lambda^2}{2}\int_{(\mathcal{I}\times\mathbb{R}^3)}\frac{id^4k}{(2\pi)^4}\,\frac{e^{-H(\sigma k^2)}}{k^2-m^2+i\varepsilon}\,\frac{e^{-H\left(\sigma(k-p)^2\right)}}{(k-p)^2-m^2+i\varepsilon}.$$

(26)

We also note that we could reduce the number of variables by setting $\vec{p}=\overrightarrow{p_1}+\overrightarrow{p_2}=\ \vec{p}_3+\vec{p}_4=0$ thanks to Lorentz invariance, so that $\mathcal{M}$ would depend only on the total external energy $p^0=p_1^0+p_2^0=p_3^0+p_4^0$

. Although this might be convenient, as it simplifies the formulas and enlightens the branch-cut nature of the singularity of $\mathcal{M}$, we will not make this assumption here, as this would veil the full dependence from the 4-momenta in the Cutkosky rules.

> 我们还注意到，根据洛伦兹不变性，我们可以通过设置 $\vec{p} = \overrightarrow{p_1} + \overrightarrow{p_2} = \vec{p}_3 + \vec{p}_4 = 0$ 减少变量数量，这样 $\mathcal{M}$ 将仅依赖于总外部能量 $p^0 = p_1^0 + p_2^0 = p_3^0 + p_4^0$。尽管这可能很方便——能简化公式并阐明 $\mathcal{M}$ 奇点的分支切割性质，但我们在这里不做这个假设，因为这会掩盖 Cutkosky 规则中四动量的完整依赖关系。

We can show that the amplitude (23) is non-singular as long as p^0 is purely imaginary. First, we note that the nonlocality in (23) is encompassed in the vertex functions. As the nonlocal form factor $e^{-H(z)}$ is an entire function with no zeroes, the vertices $\mathcal{V}$ have no poles (nor zeroes) for any p and k . That means that the singularities of the integrand in (23) are the same as those of the local scalar theory, and they correspond to the poles of the two propagators. The poles of the first propagator are obtained by imposing that the loop momentum k is on-shell, that is,

> 我们可以证明，只要 p^0 为纯虚数，振幅 (23) 就非奇异。首先，我们注意到式 (23) 中的非局域性包含在顶点函数中。由于非局域形状因子 $e^{-H(z)}$ 是没有零点的整函数，对任意 p 和 k ，顶点 $\mathcal{V}$ 都没有极点 (也没有零点)。这意味着式 (23) 中被积函数的奇点和局域标量理论的奇点相同，都对应两个传播子的极点。第一个传播子的极点可通过要求圈动量 k 在壳得到，即

$$k^2 - m^2 + i\varepsilon = \left(k^0\right)^2 - \left(\vec{k}\right)^2 = 0, \tag{27}$$

corresponding, up no negligible $O\left(\varepsilon^2\right)$ terms, to the following two values of k^0

> 对应于 k^0 的如下两个值，不存在不可忽略的 $O\left(\varepsilon^2\right)$ 项

$$\bar{k}^0_{1,2} = \pm\sqrt{\left(\vec{k}\right)^2 + m^2 - i\varepsilon}. \tag{28}$$

Such poles do not depend on the external energies, and they are always far from the imaginary axis, where the integration in k^0 is performed, as displayed in Fig. 2. The poles of the second propagator are obtained by solving

> 这些极点不依赖于外部能量，且始终远离虚轴——k^0 中的积分正是在虚轴上进行的，如图 2 所示。第二个传播子的极点可通过求解得到

$$(k-p)^2 - m^2 + i\varepsilon = \left(k^0 - p^0\right)^2 - \left(\vec{k} - \vec{p}\right)^2 - m^2 + i\varepsilon = 0, \tag{29}$$

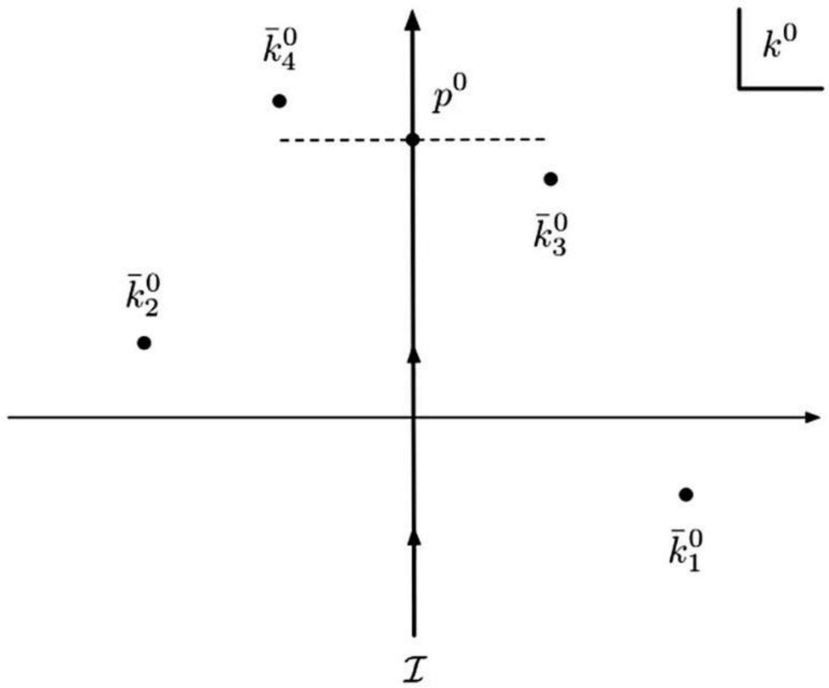
which implies that the second internal momentum $p - k$ is on-shell. This gives

> 这意味着第二个内部动量 $p - k$ 在壳上。由此得到

$$\bar{k}^0_{3,4} = p^0 \pm \sqrt{\left(\vec{k} - \vec{p}\right)^2 + m^2 - i\varepsilon}. \tag{30}$$

Fig. 2 We plot the poles $\bar{k}_1^0, \bar{k}_2^0, \bar{k}_3^0, \bar{k}_4^0$ on the complex k^0 plane, when p^0 is purely imaginary

图 2 当 p^0 为纯虚数时，我们绘制了复 k^0 平面上的极点 $\bar{k}_1^0, \bar{k}_2^0, \bar{k}_3^0, \bar{k}_4^0$



These two poles are at the right and left sides of p^0 , which is initially purely imaginary, so that they are far from $\mathcal{I}$, see Fig. 2. We conclude that the integral in (23) has no singularities for p^0 purely imaginary. In fact, the poles (28) and (30) are far from the integration contour, and the Euclidean amplitude (23) is convergent in the ultraviolet, because the conditions (17)-(18) ensure that the integrand falls to zero faster than k^{-4} for $k^2 \to -\infty$.

这两个极点分别位于初始为纯虚数的 p^0 的左右两侧，因此它们远离 $\mathcal{I}$ ，见图 2。我们得出结论: 当 p^0 为纯虚数时，(23) 式中的积分没有奇点。实际上，极点 (28) 和 (30) 远离积分围道，且欧几里得振幅 (23) 在紫外是收敛的，因为条件 (17)-(18) 确保了当 $k^2 \to -\infty$ 时，被积函数比 k^{-4} 更快地趋近于零。

We emphasize that the physical scattering amplitude for the process under consideration is obtained from (23) or (26) in the limit $\varepsilon \to 0$, that is,

我们强调，所研究过程的物理散射振幅是在极限 $\varepsilon \to 0$ 下由 (23) 或 (26) 得到的，即:

$$\mathcal{M}(p_h) = \lim_{\varepsilon \to 0} \mathcal{M}(p_h, \varepsilon), \tag{31}$$

and this limit has to be taken after we have analytically continued $\mathcal{M}(p_h, \varepsilon)$ to real external energies. This analytic continuation is achieved by sending the external energies to real and positive values $p_1^0 \to E_1 \in \mathbb{R}_0^+, p_2^0 \to E_2 \in \mathbb{R}_0^+, p_3^0 \to E_3 \in \mathbb{R}_0^+, p_4^0 \to E_4 \in \mathbb{R}_0^+$, so that $p^0 \to E \in \mathbb{R}_0^+$. Indeed, we can imagine to move p^0 continuously to the physical energy E . In doing so, the two poles $\bar{k}_{3,4}^0$ in (30) translate to the right together

p^0 , and, when $\Re\{p^0\} > 0$, it happens that the pole $\bar{k}_4^0$ passes through the imaginary axis for some values of the loop momenta $\vec{k}$ (in our notation $\Re\{a\}$ is the real part of $a \in \mathbb{C}$). Therefore, the analytic continuation of (23) is obtained by deforming continuously the integration contour in order to avoid the pole $\bar{k}_4^0$. At the end of this procedure, one has $p^0 = E$, and the pole $\bar{k}_4^0$ is close to the real axis, still with a positive infinitesimal imaginary part due to the Feynman factor $i\varepsilon$. Indeed, the analytical continuation of the amplitude (23) will be given by

且该极限必须在我们将 $\mathcal{M}(p_h, \varepsilon)$ 解析延拓到实外部能量之后取。这种解析延拓通过将外部能量移至正实数值 $p_1^0 \to E_1 \in \mathbb{R}_0^+, p_2^0 \to E_2 \in \mathbb{R}_0^+, p_3^0 \to E_3 \in \mathbb{R}_0^+, p_4^0 \to E_4 \in \mathbb{R}_0^+$ 实现，从而得到 $p^0 \to E \in \mathbb{R}_0^+$ 。事实上，我们可以设想将 p^0 连续移动到物理能量 E 。在此过程中，(30) 式中的两个极点 $\bar{k}_{3,4}^0$ 会和 p^0 一同向右移动，当 $\Re\{p^0\} > 0$ 时，对于特定的圈动量 $\vec{k}$ 值，极点 $\bar{k}_4^0$ 会穿过虚轴 (在我们的记号中，$\Re\{a\}$ 是 $a \in \mathbb{C}$ 的实部)。因此，(23) 式的解析延拓可通过连续变形积分围道以避开极点 $\bar{k}_4^0$ 得到。该过程结束后，可得 $p^0 = E$ ，且由于费曼因子 $i\varepsilon$ ，极点 $\bar{k}_4^0$ 靠近实轴，仍带有一个正无穷小虚部。实际上，振幅 (23) 的解析延拓可表示为

$$\begin{aligned}\mathcal{M}(p_h, \varepsilon) &= -\frac{\lambda^2}{2}\int_{(\mathcal{C}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{\mathcal{V}(p_1, p_2, k, p-k)}{k^2 - m^2 + i\varepsilon} \frac{\mathcal{V}(p-k, k, p_3, p_4)}{(k-p)^2 - m^2 + i\varepsilon} \\ &\equiv -\frac{\lambda^2}{2}\int_{(\mathcal{C}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{F_1(k, p_h)}{k^2 - m^2 + i\varepsilon},\end{aligned} \tag{32}$$

where $p^0 = E \in \mathbb{R}_0^+$, and the integration contour $\mathcal{C}$ in the k^0 variable is obtained by deforming the imaginary axis $\mathcal{I}$ around the pole $\bar{k}_4^0$. This picture is represented schematically in Fig. 3. In order to write the loop integrals in a compact form, in the last line of (32), we have also implicitly defined the following function:

其中 $p^0 = E \in \mathbb{R}_0^+$ ，k^0 变量中的积分围道 $\mathcal{C}$ 是通过变形极点 $\bar{k}_4^0$ 周围的虚轴 $\mathcal{I}$ 得到的。该过程的示意图如图 3 所示。为了将圈积分写为紧凑形式，我们还在 (32) 的最后一行隐式定义了如下函数:

$$F_1(k, p_h) \equiv \frac{B(k, p_h)}{(k-p)^2 - m^2 + i\varepsilon},$$

where

其中

(33)

$$B(k, p_h) \equiv \mathcal{V}(p_1, p_2, k, p-k)\,\mathcal{V}(p-k, k, p_3, p_4)\,.$$

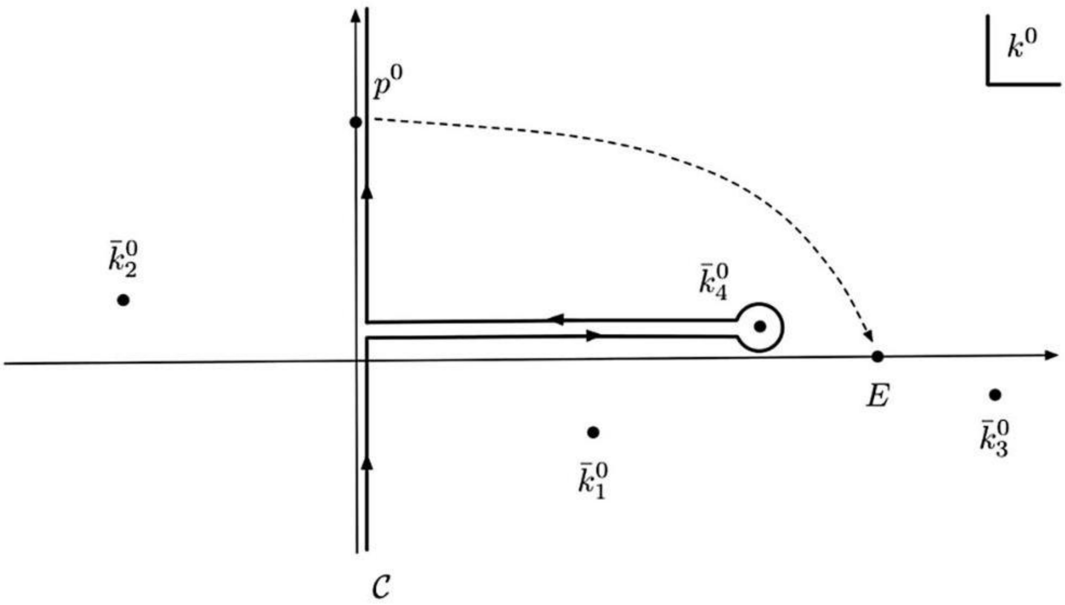


Fig. 3 We plot the poles $\bar{k}_1^0, \bar{k}_2^0, \bar{k}_3^0, \bar{k}_4^0$ for p^0 real and positive. Since $\bar{k}_1^0$ and $\bar{k}_2^0$ do not depend on p^0 , their positions do not change when $p^0 \to E$. On the contrary, $\bar{k}_3^0$ and $\bar{k}_4^0$ translate with p^0 , and $\bar{k}_4^0$ passes through the imaginary axis $\mathcal{I}$ for some values of $\vec{k}$. We also plot the contour $\mathcal{C}$, which is obtained by deforming $\mathcal{I}$ around $\bar{k}_4^0$

图 3 我们绘制了 $\bar{k}_1^0, \bar{k}_2^0, \bar{k}_3^0, \bar{k}_4^0$ 的极点，其中 p^0 为正实数。由于 $\bar{k}_1^0$ 和 $\bar{k}_2^0$ 不依赖于 p^0 ，当 $p^0 \to E$ 时它们的位置不发生改变。相反，$\bar{k}_3^0$ 和 $\bar{k}_4^0$ 随 p^0 平移，对某些 $\vec{k}$ 取值，$\bar{k}_4^0$ 会穿过虚轴 $\mathcal{I}$ 。我们还绘制了围道 $\mathcal{C}$ ，它是将 $\mathcal{I}$ 沿 $\bar{k}_4^0$ 周边形变得到的

By means of the procedure of analytic continuation described above, one can show that (23) is still nonsingular for $E \in \mathbb{R}_0^+$ as far as the term $i\varepsilon$ is maintained. This fact can be understood intuitively as follows: If $\varepsilon > 0$, all the poles (28) and (30) are well separated. In particular, the poles $\bar{k}_1^0$ and $\bar{k}_4^0$ have a small negative and positive complex part, respectively, and the deformed contour $\mathcal{C}$ passes between them, but it never touches them. This means that the integrand in (32) has no poles in the integration domain, which implies that the complex amplitude (32) is nonsingular for any real E . However, the $i\varepsilon$ term is just an artifact used to regularize (23), and it must be sent to zero to recover the physical scattering amplitude. Yet, when $\varepsilon \to 0$ and $E \geq 2m$, the poles $\bar{k}_1^0$ and $\bar{k}_4^0$ coincide for some values of the loop spatial momentum $\vec{k}$. In this situation, the integration contour is pinched by these poles, and it cannot be deformed further. As we will show, this implies that the complex amplitude develops a branch-cut singularity at $E \geq 2m$. We just mention that the scattering amplitude has other singularities at $E = 0$ and $E \leq -2m$, which we neglect as they are unphysical. In fact, the external energy is always positive for massive particles, and in the jargon of quantum field theory, it is said that these singularities are not on the physical sheet, see [28].

借助上述解析延拓步骤可以证明，只要保留项 $i\varepsilon$ ，式 (23) 对 $E \in \mathbb{R}_0^+$ 仍然非奇异。这一点可以通过直观解释如下: 如果 $\varepsilon > 0$ ，所有极点 (28) 和 (30) 都完全分离。具体来说，极点 $\bar{k}_1^0$ 和 $\bar{k}_4^0$ 分别带有小幅的负复部和正复部，形变后的围道 $\mathcal{C}$ 从二者之间穿过，且从不接触它们。这说明式 (32) 的被积函数在积分区域内没有极点，即复振幅 (32) 对任意实数 E 都非奇异。但 $i\varepsilon$ 项只是用于正则化式 (23) 的人为引入项，必须将其取零才能得到物理散射振幅。然而，当 $\varepsilon \to 0$ 且 $E \geq 2m$ 时，对圈空间动量 $\vec{k}$ 的某些取值，极点 $\bar{k}_1^0$ 和 $\bar{k}_4^0$ 重合。这种情况下，积分围道被这两个极点夹住，无法进一步形变。我们后续会说明，这意味着复振幅在 $E \geq 2m$ 处产生了割线奇点。这里仅提及散射振幅在 $E = 0$ 和 $E \leq -2m$ 处还存在其他奇点，由于它们是非物理的我们将其忽略。实际上，对有质量粒子而言外部能量始终为正，用量子场论的行话来说，这些奇点不在物理叶上，参见文献 [28]。

As we will show below by direct calculation, this picture, which might appear a little bit naive at first look, is correct, and it is valid for generic diagrams. We stress that when the poles $\bar{k}_1^0$ and $\bar{k}_4^0$ coincide, the two propagators in (23) are on-shell for the same value of the loop momenta k . Therefore, in order to find the singularities of complex amplitudes, one should impose that two (or more in the case of generic amplitudes) of the internal momenta are on-shell at the same time. In our case, after setting $\varepsilon = 0$, we have to solve simultaneously (27) and (29). This system of equations is usually referred to as Landau equations, see [28] for review. For completeness, we mention that Landau equations also include the condition that the on-shell momenta must be collinear. However, in our case (27) and (29) encompass this condition, as they imply that $p = 2k$.

我们后续会通过直接计算证明，这个乍看略显朴素的图像是正确的，且对任意费曼图都成立。我们要强调，当极点 $\bar{k}_1^0$ 和 $\bar{k}_4^0$ 重合时，式 (23) 中的两个传播子对圈动量 k 的同一个值都在壳。因此，要寻找复振幅的奇点，应当要求两个 (对一般振幅可以更多) 内动量同时在壳。在我们的问题中，取 $\varepsilon = 0$ 后，我们需要联立求解方程 (27) 和 (29)。这个方程组通常被称为朗道方程，综述参见文献 [28]。为完整起见，我们说明朗道方程还包含在壳动量必须共线的条件。但在我们的问题中，方程 (27) 和 (29) 已经包含了这个条件，因为它们给出 $p = 2k$ 。

At that point, we are ready to evaluate (32). From Fig. 3, we observe that the integral along the contour $\mathcal{C}$ receives two contributions: The first is given integrating along $\mathcal{I}$, and the second along the small circle surrounding $\bar{k}_4^0$. This second part is easily calculated using the residues theorem, so that we get

至此，我们已经可以计算式 (32)。从图 3 中可以看出，沿围道 $\mathcal{C}$ 的积分由两部分贡献组成: 第一部分是沿 $\mathcal{I}$ 积分得到，第二部分是沿环绕 $\bar{k}_4^0$ 的小圆积分得到。第二部分可以用留数定理轻松计算，最终得到

$$\mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^2}{2}\left[\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{F_1(k)}{k^2 - m^2 + i\varepsilon} \right.$$
$$\left. + \int_{(\mathbb{R}^3)} \frac{id^3k}{(2\pi)^4} (-2\pi i) \left(\frac{\sigma\left(\Re\left\{k^0\right\}\right)}{2\sqrt{\left(\vec{k} - \vec{p}\right)^2 + m^2 - i\varepsilon}} \frac{B(k, p_h)}{k^2 - m^2 + i\varepsilon} \right) \Bigg|_{k^0 = \bar{k}_4^0}^{0} \right],$$
(34)

where $\sigma(x)$ is the Heaviside step function

其中 $\sigma(x)$ 是海维赛德阶跃函数

$$\sigma(x) = \begin{cases} 1, & x > 0 \\ 1/2, & x = 0 \\ 0, & x < 0 \end{cases} \tag{35}$$

The term $\sigma\left(\Re\left(\bar{k}_4^0\right)\right)$ in (34) implies that when $\Re\left(\bar{k}_4^0\right) > 0$ and the pole $\bar{k}_4^0$ is in the first quadrant of the complex plane, indeed it crosses the imaginary axis when we send $p^0 \to E$, and the integral along the small circle contributes a $2\pi i$ times the residue of the integrand at $\bar{k}_4^0$. On the contrary, when $\Re\left(\bar{k}_4^0\right) < 0$ and $\bar{k}_4^0$ is in the second quadrant, the integration contour is not deformed, and we only have the contribution from $\mathcal{I}$. Moreover, when the pole $\bar{k}_4^0$ is on the imaginary axis, so that $\Re\left(\bar{k}_4^0\right) = 0$, and the integral over $\mathcal{C}$ is given by the integral along the imaginary axis (intended as the principal part), plus a contribution from a small semicircle centered at $\bar{k}_4^0$. This is why the Heaviside function (35) must be equal to 1/2 at $x = 0$ to give the correct result. However, $\Re\left(\bar{k}_4^0\right) = 0$ on a subset of $\vec{k} \in \mathbb{R}^3$ with zero measure; indeed the contribution of the small semicircle gives zero when integrated in d^3k .

式 (34) 中的项 $\sigma\left(\Re\left(\bar{k}_4^0\right)\right)$ 表明，当 $\Re\left(\bar{k}_4^0\right) > 0$ 且极点 $\bar{k}_4^0$ 位于复平面第一象限时，当我们取 $p^0 \to E$ 时，极点确实会穿过虚轴，此时沿小圆的积分贡献为 $2\pi i$ 乘以被积函数在 $\bar{k}_4^0$ 处的留数。反之，当 $\Re\left(\bar{k}_4^0\right) < 0$ 且 $\bar{k}_4^0$ 位于第二象限时，积分围道不会发生形变，我们仅得到来自 $\mathcal{I}$ 的贡献。此外，当极点 $\bar{k}_4^0$ 位于虚轴上时，满足 $\Re\left(\bar{k}_4^0\right) = 0$ ，此时对 $\mathcal{C}$ 的积分等于沿虚轴的主值积分，加上一个以 $\bar{k}_4^0$ 为中心的小半圆的贡献。这就是为什么海维赛德函数 (35) 在 $x = 0$ 处必须等于 1/2 才能得到正确结果。不过，$\Re\left(\bar{k}_4^0\right) = 0$ 仅发生在 $\vec{k} \in \mathbb{R}^3$ 的一个零测度子集上；实际上，对 d^3k 积分时，小半圆的贡献为零。

Let us concentrate on the second term in (34). Since we are interested in the limit $\varepsilon \to 0$ of $\mathcal{M}(p_h, \varepsilon)$, and $\left(\vec{k} - \vec{p}\right)^2 + m^2 > 0$, we can ignore the $i\varepsilon$ term in $\sqrt{\left(\vec{k} - \vec{p}\right)^2 + m^2 - i\varepsilon}$. Then, we can make use of the following relation:

我们来重点讨论式 (34) 中的第二项。由于我们关心 $\mathcal{M}(p_h, \varepsilon)$ 取 $\varepsilon \to 0$ 的极限，且已知 $\left(\vec{k} - \vec{p}\right)^2 + m^2 > 0$ ，因此可以忽略 $\sqrt{\left(\vec{k} - \vec{p}\right)^2 + m^2 - i\varepsilon}$ 中的 $i\varepsilon$ 项。接下来我们可以利用下述关系：

$$\int_{\mathbb{R}^3} d^3k \left. \frac{f(k,h)}{2\sqrt{\left(\vec{k} - \vec{h}\right)^2 + m^2}} \right|_{k^0 \equiv h^0 \mp \sqrt{\left(\vec{k} - \vec{h}\right)^2 + m^2}}$$
$$= \int_{\mathbb{R}^4} d^4k \sigma\left(\pm\left(h^0 - k^0\right)\right) \delta\left((h-k)^2 - m^2\right) f(k,h) \tag{36}$$

that can be easily verified by the reader, so that we have

读者可以自行验证该式，由此我们得到

$$\int_{(\mathbb{R}^3)} d^3k \left(\frac{\sigma\left(\Re\left\{k^0\right\}\right)}{2\sqrt{\left(\vec{k}-\vec{p}\right)^2+m^2-i\varepsilon}} \frac{B\left(k,p_h\right)}{k^2-m^2+i\varepsilon} \right)\Bigg|_{k^0=\vec{k}_4^0}$$

$$= \int_{(\mathbb{R}^4)} d^4k \frac{B\left(k,p_h\right)}{k^2-m^2+i\varepsilon} \sigma\left(k^0\right)\sigma\left(p^0-k^0\right)\delta\left((p-k)^2-m^2\right), \tag{37}$$

where we have maintained the term $i\varepsilon$ in the propagator of the internal line of 4- momentum k , since k can still go on-shell. Replacing (37) with (34), we have

其中我们在 4-动量 k 的内线传播子中保留了项 $i\varepsilon$，因为 k 仍可以在壳。将 (37) 代入 (34)，我们得到

$$\mathcal{M}\left(p_h,\varepsilon\right) = -\frac{\lambda^2}{2}\left[\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} B\left(k,p_h\right) \frac{1}{k^2-m^2+i\varepsilon} \frac{1}{(k-p)^2-m^2+i\varepsilon}\right.$$

$$\left. + \int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} \frac{B\left(k,p_h\right)}{k^2-m^2+i\varepsilon} (-2\pi i)\,\sigma\left(k^0\right)\sigma\left(p^0-k^0\right)\delta\left((p-k)^2-m^2\right)\right].$$

(38)

Equation (38) is almost our final result. It is the analytic continuation of the Euclidean scattering amplitude (23) to real positive external energies $p^0 = E \in \mathbb{R}_0^+$. The next step is to evaluate the imaginary part of (38) in the limit $i\varepsilon \to 0$.

式 (38) 基本就是我们的最终结果。它是欧几里得散射振幅 (23) 解析延拓到实正外部能量 $p^0 = E \in \mathbb{R}_0^+$ 的结果。下一步我们需要在 $i\varepsilon \to 0$ 的极限下计算式 (38) 的虚部。

Let us show that the first integral in (38) is real for $i\varepsilon \to 0$, and indeed it does not contribute to $\mathcal{M}-\mathcal{M}^*$. We note that the poles (28) are far from the integration contour, see Fig. 3. In fact, as $k \in \mathcal{I}\times\mathbb{R}^3$, so that $k^2-m^2>0$, and one can neglect the $i\varepsilon$ term in $k^2-m^2+i\varepsilon$. Moreover, $p \in \mathbb{R}_0^+\times\mathbb{R}^3$; indeed the term $(k-p)^2-m^2$ can be zero only for $k^0=0$. Thus, we can neglect the $i\varepsilon$ term in $(k-p)^2-m^2+i\varepsilon$, except on the subset $k^0=0$ of the integration domain. However, such a subset has zero measure, and the contribution of this region is zero when integrated in d^4k . We can prove this fact more explicitly, noting that

我们来证明 (38) 中的第一个积分对 $i\varepsilon \to 0$ 是实的，且它确实对 $\mathcal{M}-\mathcal{M}^*$ 没有贡献。我们注意到极点 (28) 远离积分轮廓，参见图 3。实际上，当 $k \in \mathcal{I}\times\mathbb{R}^3$ 时，可得 $k^2-m^2>0$ ，因此可以忽略 $k^2-m^2+i\varepsilon$ 中的 $i\varepsilon$ 项。此外，$p \in \mathbb{R}_0^+\times\mathbb{R}^3$ ；事实上仅当 $k^0=0$ 时项 $(k-p)^2-m^2$ 才能为零。因此，除了积分域中的子集 $k^0=0$ 外，我们可以忽略 $(k-p)^2-m^2+i\varepsilon$ 中的 $i\varepsilon$ 项。但该子集的测度为零，在对 d^4k 积分时该区域的贡献为零。我们可以更明确地证明这一点，注意到

$$\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{k^2-m^2+i\varepsilon} \frac{B\left(k,p_h\right)}{(k-p)^2-m^2+i\varepsilon}$$

$$= \int_{\mathcal{I}} idk^0 \int_{\mathbb{R}^3} \frac{d^3k}{k^2-m^2} \frac{B\left(k,p_h\right)}{(k-p)^2-m^2+i\varepsilon}$$

$$= \int_{\mathcal{I}} idk^0 \int_{\mathbb{R}^3} \frac{d^3h}{\left(k^0\right)^2-\left(\vec{h}+\vec{p}\right)^2-m^2} \frac{B\left(k,p_h\right)}{\left(k^0-p^0\right)^2-\vec{h}^2-m^2+i\varepsilon}$$

$$= \int_{\mathcal{J}} idk^0 \int \frac{h^2 dh d\Omega}{\left(k^0\right)^2 - \vec{p}^2 - h\left(h - 2\left|\vec{p}\right|\cos\left(\theta\right)\right) - m^2} \frac{B\left(k, p_h\right)}{\left(k^0 - p^0\right)^2 - h^2 - m^2 + i\varepsilon}.$$

(39)

In (39) we have neglected the $i\varepsilon$ term in the first propagator, as explained before. Moreover, we have used the change of variable $\vec{k} = \vec{h} + \vec{p}$, and we switched to polar coordinates. Let us consider the following formula:

如前所述，我们在 (39) 的第一个传播子中忽略了 $i\varepsilon$ 项。此外，我们做了变量替换 $\vec{k} = \vec{h} + \vec{p}$，并转换为极坐标。我们来看下述公式:

$$\lim_{\varepsilon \to 0} \int \frac{f\left(x\right)}{x^2 - a^2 \pm i\varepsilon} dx = \mathcal{P} \int \frac{f\left(x\right)}{x^2 - a^2} dx \mp i\pi \int f\left(x\right) \delta\left(x^2 - a^2\right) dx \tag{40}$$

where $\mathcal{P}$ is the principal part of the integral, which can be derived from the residue theorem. One can use (40) to perform the h integration in (39), so that

其中 $\mathcal{P}$ 是积分的主值，可以由留数定理推导得到。利用 (40) 即可完成 (39) 中 h 的积分，得到

$$\lim_{\varepsilon \to 0} \int_{\left(\mathcal{J} \times \mathbb{R}^3\right)} \frac{id^4k}{k^2 - m^2 + i\varepsilon} \frac{B\left(k, p_h\right)}{\left(k - p\right)^2 - m^2 + i\varepsilon} = \mathcal{P} \int_{\left(\mathcal{J} \times \mathbb{R}^3\right)} \frac{id^4k}{k^2 - m^2} \frac{B\left(k, p_h\right)}{\left(k - p\right)^2 - m^2}$$
$$+ \int_{\left(\mathcal{J} \times \mathbb{R}^3\right)} \frac{id^4k}{k^2 - m^2} \frac{B\left(k, p_h\right)}{\left(k - p\right)^2 - m^2} \left(-i\pi\right) \delta\left(\left(k - p\right)^2 - m^2\right) S\left(k^0\right), \tag{41}$$

where

其中

$$S\left(x\right) = \begin{cases} 1 & \text{for} \quad x = 0 \\ 0 & \text{for} \quad x \neq 0 \end{cases} \tag{42}$$

We stress that the $S\left(k^0\right)$ in (41) is needed since $\left(k^0 - p^0\right)^2 - h^2 - m^2$ can be zero only for $k^0 = 0$. Furthermore, the last integral in (41) is zero, as the integrand has support on the subset $\left\{k^0 = 0; \left(k - p\right)^2 = m^2\right\}$ of the integration domain, which has null measure. Finally, the remaining integral in (41) is real. In fact, its complex conjugate is

我们需要强调，(41) 中的 $S\left(k^0\right)$ 是必要的，因为仅当 $k^0 = 0$ 时 $\left(k^0 - p^0\right)^2 - h^2 - m^2$ 才能为零。此外，(41) 中的最后一个积分为零，因为被积函数的支集在积分域的子集 $\left\{k^0 = 0; \left(k - p\right)^2 = m^2\right\}$ 上，而该子集测度为零。最后，(41) 中剩余的积分是实的。实际上，它的复共轭为

$$\left(\mathcal{P} \int_{\mathcal{J}} idk^0 \int_{\mathbb{R}^3} \frac{d^3k}{k^2 - m^2} \frac{B\left(k, p_h\right)}{\left(k - p\right)^2 - m^2}\right)^*$$
$$= \left(\mathcal{P} \int_{-\infty}^{+\infty} i^2 dz^0 \int_{\mathbb{R}^3} \frac{d^3k}{\left(iz\right)^2 - \vec{k}^2 - m^2} \frac{B\left(iz, \vec{k}, p_h\right)}{\left(iz - p^0\right)^2 - \left(\vec{k} - \vec{p}\right)^2 - m^2}\right)^*$$

$$= \mathcal{P}\int_{-\infty}^{+\infty} i^2 dz^0 \int_{\mathbb{R}^3} \frac{d^3k}{(-iz)^2 - \vec{k}^2 - m^2} \frac{B\left(-iz, \vec{k}, p_h\right)}{(-iz - p^0)^2 - \left(\vec{k} - \vec{p}\right)^2 - m^2}$$

$$= \mathcal{P}\int_{-\infty}^{+\infty} i^2 dz^0 \int_{\mathbb{R}^3} \frac{d^3k}{(iz)^2 - \vec{k}^2 - m^2} \frac{B\left(iz, \vec{k}, p_h\right)}{(iz - p^0)^2 - \left(\vec{k} - \vec{p}\right)^2 - m^2}, \tag{43}$$

where we have defined $k^0 = iz$ and used the redefinition $z \to -z$ in the last equality. Thus, the first integral in (38) is real. Moreover, its integrand has no poles for $k^0 \neq 0$ that is everywhere but on a subset of zero measures, so that it is also non-singular.

这里我们定义了 $k^0 = iz$，并在最后一个等号中做了重定义 $z \to -z$。因此，(38) 中的第一个积分是实的。此外，除了一个零测度子集外，对所有 $k^0 \neq 0$ 被积函数都没有极点，因此它也是非奇异的。

So far, we have proved that

到目前为止，我们已经证明了

$$\lim_{\varepsilon\to 0} \mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^2}{2}\left[\mathcal{P}\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} B(k, p_h) \frac{1}{k^2 - m^2} \frac{1}{(k-p)^2 - m^2}\right.$$
$$\left. + \lim_{\varepsilon\to 0}\int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} \frac{B(k, p_h)}{k^2 - m^2 + i\varepsilon} (-2\pi i)\,\sigma(k^0)\,\sigma(p^0 - k^0)\,\delta\left((p-k)^2 - m^2\right)\right],$$

(44)where the first term is real and non-singular. We can now rearrange the second integral using (40), so that

其中第一项是实的且非奇异。我们现在可以利用 (40) 整理第二个积分，得到

$$\lim_{\varepsilon\to 0} \mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^2}{2}\left[\mathcal{P}\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} B(k, p_h) \frac{1}{k^2 - m^2} \frac{1}{(k-p)^2 - m^2}\right.$$
$$+ \mathcal{P}\int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} \frac{B(k, p_h)}{k^2 - m^2} (-2\pi i)\,\sigma(k^0)\,\sigma(p^0 - k^0)\,\delta\left((p-k)^2 - m^2\right)$$
$$\left. + \int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} B(k, p_h)\left(2\pi^2 i^2\right)\sigma(p^0 - k^0)\,\delta\left((p-k)^2 - m^2\right)\sigma(k^0)\,\delta\left(k^2 - m^2\right)\right].$$

(45)

This is the final form of the scattering amplitude of the one-loop diagram under consideration. According to this expression, the imaginary part of the amplitude is

这就是我们所讨论的单圈图散射振幅的最终形式。根据该表达式，振幅的虚部为

$$\lim_{\varepsilon\to 0}\left(\mathcal{M}(p_h, \varepsilon) - \mathcal{M}(p_h, \varepsilon)^*\right)$$

$$= -\frac{\lambda^2}{2}\int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} B(k, p_h)(-2\pi i)^2 \sigma(p^0 - k^0)\,\delta\left((p-k)^2 - m^2\right)\sigma(k^0)\,\delta\left(k^2 - m^2\right).$$

(46)

Finally, we note that we can set $B(k,p_h)=1$ in (46). In fact, all momenta are on-shell, since p_1 and p_2 are the 4-momenta of the external particles, while k and $p-k$ are on-shell due to the delta functions. From the definitions (24) and (33), one has

最后，我们注意到可以令 (46) 中 $B(k,p_h)=1$ 。实际上，所有动量都在壳上，因为 p_1 和 p_2 是外部粒子的四维动量，而 k 和 $p-k$ 因 δ 函数的作用也满足在壳条件。根据定义 (24) 和 (33)，可得

$$B(k,p_h)\,\delta\left((p-k)^2-m^2\right)\delta\left(k^2-m^2\right)$$

$$=e^{-H\left(\sigma(p_1)^2\right)}e^{-H\left(\sigma(p_2)^2\right)}e^{-H\left(\sigma(k)^2\right)}e^{-H\left(\sigma(p-k)^2\right)}\delta\left((p-k)^2-m^2\right)\delta\left(k^2-m^2\right),$$

(47)

and using (2), it is now evident that we can replace $B(k,p_h)=1$ in (46). Therefore, the imaginary part of $\mathcal{M}$ is

再利用 (2)，显然我们可以将 (46) 中的 $B(k,p_h)=1$ 替换。因此，$\mathcal{M}$ 的虚部为

$$\mathcal{M}(p_h)-\mathcal{M}(p_h)^*$$

$$=-\frac{\lambda^2}{2}\int_{(\mathbb{R}^4)}\frac{id^4k}{(2\pi)^4}(-2\pi i)^2\sigma\left(p^0-k^0\right)\delta\left((p-k)^2-m^2\right)\sigma\left(k^0\right)\delta\left(k^2-m^2\right).$$

(48)

From (48) it is clear that $\mathcal{M}$ is real for $E<2m$, while its imaginary part has a jump at $E=2m$, as it is nonzero for $E\geq 2m$, when both k and $p-k$ can be on-shell at the same time. Moreover, from (45) one can also prove that $\mathcal{M}$ has a branch-cut singularity at $E\geq 2m$. In fact, the singular part of the amplitude comes from the last two integrals in (45), when the internal 4-moments are on-shell. In proximity of these values, the function $B(k,p_h)$ is practically one, as we have already noted, and indeed the singularity of the nonlocal amplitude will be the same as that of the local scalar field theory, which is a branch-cut singularity.

由 (48) 可知，对于 $E<2m$ ，$\mathcal{M}$ 为实数，而其虚部在 $E=2m$ 处存在跃变: 这是因为当 $E\geq 2m$ 时，k 和 $p-k$ 可同时满足在壳条件，因此虚部非零。此外，由 (45) 还可证明 $\mathcal{M}$ 在 $E\geq 2m$ 处存在割线奇点。实际上，振幅的奇异部分来自 (45) 中最后两个积分，对应内四动量满足在壳条件的情况。如我们此前指出，在这些取值附近，函数 $B(k,p_h)$ 近似为 1，因此非定域振幅的奇点与定域标量场论一致，均为割线奇点。

Let us resume the main results of this section. First, we started with the complex amplitude (23) in Euclidean space, that is, for purely imaginary p^0 . Then, we have defined a procedure to continue analytically this expression moving p^0 to real values and deforming the integration contour in the k^0 variable. We have found that, in the limit $\varepsilon\to 0$,(23) is singular for values of the external momenta such that the two propagators can be simultaneously on-shell. Solving the Landau equations (27,29), and by direct inspection of the poles of the propagators, we have found that $\mathcal{M}$ has a branch-cut singularity at $E\geq 2m$. Then, we have calculated the imaginary part of $\mathcal{M}$, obtaining (48). This expression shows that $\mathcal{M}$ is real, unless the two propagators are on-shell, otherwise the two delta functions would be zero, and this is exactly the condition expressed by the

Landau equations. This implies that the imaginary part of $\mathcal{M}$ is nonzero only along the branch-cut singularity, that is, for $E \geq 2m$. We emphasize once more that since the nonlocal propagators have the same poles of the local theory, as the nonlocal form factor is an entire function with no zeroes, the nonlocal scattering amplitude (23) has the same singularity structure of the corresponding amplitude in the local scalar theory. This is a general property of the nonlocal theory, as will be discussed in section ”Generic Amplitudes”.

我们概括一下本节的主要结论。首先，我们从欧几里得空间中的复振幅 (23) 出发，该情况对应 p^0 为纯虚数。随后我们定义了解析延拓方案: 将 p^0 移动到实数值，并对 k^0 变量的积分围道做形变。我们发现，在 $\varepsilon \to 0$ 极限下，当两个传播子可同时在壳时，(23) 在外动量取对应值时奇异。通过求解朗道方程 (27,29)，并直接检验传播子的极点，我们得到 $\mathcal{M}$ 在 $E \geq 2m$ 处存在割线奇点。随后我们计算了 $\mathcal{M}$ 的虚部，得到结果 (48)。该式表明，除非两个传播子同时在壳，否则 $\mathcal{M}$ 为实——两个传播子同时在壳时两个 δ 函数才非零，这正是朗道方程所表达的条件。这说明 $\mathcal{M}$ 的虚部仅沿割线奇点非零，也就是对应 $E \geq 2m$ 的情况。我们再次强调: 由于非定域传播子与定域理论的极点位置相同，且非定域形状因子是无零点的整函数，因此非定域散射振幅 (23) 与定域标量论中对应振幅的奇点结构完全一致。这是非定域理论的一般性质，我们将在“一般振幅”一节讨论。

We conclude this section by comparing (26) with (48). We see that the imaginary part of $\mathcal{M}$ is obtained by replacing each nonlocal propagator by

我们在本节最后对比 (26) 与 (48)。可以看到，$\mathcal{M}$ 的虚部可通过将每个非定域传播子替换为

$$\frac{ie^{-H(\sigma p^2)}}{p^2 - m^2 + i\varepsilon} \to (-2\pi i)\,\sigma\left(p^0\right)\delta\left(p^2 - m^2 + i\varepsilon\right), \tag{49}$$

and replacing the integration region $\mathcal{I} \times \mathbb{R}^3$ with $\mathbb{R}^4$. These prescriptions are the generalization of the Cutkosky rules to the nonlocal quantum scalar theories. In section ”Generic Amplitudes” we will show that the Cutkosky rules give the right imaginary part of $\mathcal{M}$ for all scattering processes.

并将积分区域 $\mathcal{I} \times \mathbb{R}^3$ 替换为 $\mathbb{R}^4$ 得到。这套规则是卡特斯基规则向非定域量子标量论的推广。在“一般振幅”一节我们将证明，卡特斯基规则对所有散射过程都能正确给出 $\mathcal{M}$ 的虚部。

Cutkosky Rules, Normal and Anomalous Thresholds, and Unitarity

卡特科斯基规则、正规与反常阈以及幺正性

In this section we define the Cutkosky rules for generic diagrams and discuss their relation with the unitarity of the theory. Moreover, we define the normal and anomalous thresholds. In particular, we argue that Cutkosky rules imply that the theory is unitary if, additionally, no anomalous threshold contributes to (14). The derivation of the Cutkosky rules for generic scattering processes and the proof of the unitarity of the nonlocal theory will be presented in section ”Generic Amplitudes”.

本节我们将针对一般图定义卡特科斯基规则，讨论它们与理论幺正性的关系。此外我们还会定义正规阈与反常阈。我们特别指出，若不存在反常阈对式 (14) 的额外贡献，卡特科斯基规则即可推出该理论是幺正的。一般散射过程的卡特科斯基规则推导，以及非局域理论的幺正性证明，将在“一般振幅”一节给出。

Let us consider a general scattering amplitude for a process with L -loop in the theory (1). For a generic diagram with I internal lines and V vertices, the scattering amplitude is

我们来考虑理论 (1) 中一个 L 圈过程的一般散射振幅。对一张含 I 条内线、V 个顶点的一般图，散射振幅为

$$
i\mathcal{M}(p_h,\varepsilon)=\frac{1}{S_\#}\int_{(\mathcal{I}\times\mathbb{R}^3)^I}\prod_{i=1}^{I}\frac{d^4k_i}{(2\pi)^4}\frac{i}{k_i^2-m^2+i\varepsilon}\prod_{j=1}^{V}(-i)\lambda\mathcal{V}\left(p^{(j)}\right)(2\pi)^4\delta^{(4)}\left(\sum_{\ell=1}^{N}p_\ell^{(j)}\right),
$$

(50)

where $\mathcal{I}$ is the imaginary axis, $S_\#$ is a combinatoric factor, and the $\delta^{(4)}\left(\sum_\ell p_\ell^{(j)}\right)$ gives the momentum conservation at the j -th vertex. The vertex term $\mathcal{V}\left(p_\ell^{(j)}\right)$ has been already defined in (22). After integration of the delta functions in (50), one has

其中 $\mathcal{I}$ 是虚轴，$S_\#$ 是组合因子，$\delta^{(4)}\left(\sum_\ell p_\ell^{(j)}\right)$ 给出第 j 个顶点处的动量守恒。顶点项 $\mathcal{V}\left(p_\ell^{(j)}\right)$ 已经在式 (22) 中定义过了。对式 (50) 中的 δ 函数积分后，可得

$$
\mathcal{M}(p_h,\varepsilon)=-\frac{\lambda^V}{S_\#}\int_{(\mathcal{I}\times\mathbb{R}^3)^L}\prod_{i=1}^{L}\frac{id^4k_i}{(2\pi)^4}\frac{1}{k_i^2-m^2+i\varepsilon}\prod_{j=1}^{I-L}\frac{1}{q_j^2-m^2+i\varepsilon}B(k_i,p_h),
$$

(51)

where q_j is a linear combination with coefficients ± 1 of the loop momenta k_i and of the external momenta p_h . In (51) we have used the topological relation $I-V=\ L-1$, which implies $i^{I-V}\ =i^{L-1}$, and we have defined the function $B(k_i,p_h)$ as

其中 q_j 是圈动量 k_i 和外动量 p_h 以系数 ± 1 构成的线性组合。在式 (51) 中我们用到了拓扑关系 $I-V=\ L-1$ ，该关系给出 $i^{I-V}=i^{L-1}$ ，我们还将函数 $B(k_i,p_h)$ 定义为

$$
B(k_i,p_h)\equiv\prod_{j=1}^{V}\mathcal{V}\left(p_\ell^{(j)}\right). \tag{52}
$$

Since the theory is defined in Euclidean space, in (51) we take purely imaginary values of the external energies p_h^0 , so that the poles of the propagators are far from the integration contour. Afterward, we consider the analytic continuation of (51), taking the limit in which the external energies go to their physical real values. This is obtained by deforming the integration contour $\mathcal{I}\times\mathbb{R}^3$ for each loop, around the poles that pass through the imaginary axis $\mathcal{I}$ when $p_h^0\to E_h\in\mathbb{R}_0^+$. Therefore, the analytic continuation of (51) is given by

由于该理论定义在欧几里得空间中，我们在式 (51) 中取外能 p_h^0 为纯虚数，因此传播子的极点远离积分围道。随后我们对式 (51) 做解析延拓，取外能趋于物理实数值的极限。这一步通过对每个圈的积分围道 $\mathcal{I} \times \mathbb{R}^3$ 做形变得到: 当 $p_h^0 \to E_h \in \mathbb{R}_0^+$ 时，会有极点穿过虚轴 $\mathcal{I}$ ，我们将围道形变绕开这些极点。因此式 (51) 的解析延拓为

$$\mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^V}{S_\#} \int_{(\mathcal{C}_L \times \mathbb{R}^3)} \int_{(\mathcal{C}_{L-1} \times \mathbb{R}^3)} \cdots \int_{(\mathcal{C}_1 \times \mathbb{R}^3)} \prod_{i=1}^{L} \frac{i d^4 k_i}{(2\pi)^4} \frac{1}{k_i^2 - m^2 + i\varepsilon} \times \prod_{j=1}^{I-L} \frac{1}{q_j^2 - m^2 + i\varepsilon} B(k_i, p_h) \tag{53}$$

where $\mathcal{C}_i \times \mathbb{R}^3$ is the deformed contour for the i -th integration variable.

其中 $\mathcal{C}_i \times \mathbb{R}^3$ 是第 i 个积分变量的形变围道。

The physical scattering amplitude $\mathcal{M}(p_h)$ is given by (53) in the limit $\varepsilon \to 0$, see (31). The deformation of the integration contour $\mathcal{C}$ around the poles is allowed as long as such contour is not constrained between separate poles that overlap, and this is always possible whether the Feynman $i\varepsilon$ term is maintained. On the contrary, when $\varepsilon \to 0$, some poles overlap for some real value of the external energies. When this happens, and the integration contour is constrained between these poles, $\mathcal{C}$ cannot be deformed further, so that the amplitude has a branch-cut singularity, and we say that the integration contour is pinched by the poles. This situation is represented in Fig. 3 for the one-loop amplitude (23). We see that the integration contour $\mathcal{C}$ passes between the poles $\bar{k}_1^0$ and $\bar{k}_4^0$. When $E \geq 2m$ and in the limit $\varepsilon \to 0$, these two poles overlap for some values of k , pinching $\mathcal{C}$. This implies that the amplitude (23) has a branch-cut singularity for $E \geq 2m$.

物理散射振幅 $\mathcal{M}(p_h)$ 由极限 $\varepsilon \to 0$ 下的式 (53) 给出，参见式 (31)。只要积分围道 $\mathcal{C}$ 不被夹在多个重叠的分离极点之间，就可以对围道做形变绕开极点，只要保持费恩曼 $i\varepsilon$ 项成立，这总是可以做到的。反之，当 $\varepsilon \to 0$ 时，外能取某些实数值时会出现极点重叠。这种情况下，积分围道被夹在这些极点之间，无法继续形变 $\mathcal{C}$ ，因此振幅产生割线奇点，我们称积分围道被极点夹住了。单圈振幅 (23) 的这种情况如图 3 所示。可以看到积分围道 $\mathcal{C}$ 从极点 $\bar{k}_1^0$ 和 $\bar{k}_4^0$ 之间穿过。当 $E \geq 2m$ ，在极限 $\varepsilon \to 0$ 下，对 k 的某些取值，这两个极点会发生重叠，夹住 $\mathcal{C}$ 。这说明振幅 (23) 在 $E \geq 2m$ 处存在割线奇点。

We note that the nonlocality of the theory is encoded only in the term $B(k_i, p_h)$, which ensures the ultraviolet convergence of the integrals in the scattering amplitudes. Since, by hypothesis, the nonlocal form factor is an entire function with no zeros in the finite complex plan, also $B(k_i, p_h)$ has no zeros or poles for finite momenta k_i and p_h . Thus, this function does not introduce other poles in the integrand in (51) than those of the propagators, which are the same as in the scalar local theory. Indeed, just as in the local theory, the singularities of (51) are found by solving the Landau equations, which simply express the condition that the 4- momenta of two or more internal lines are on-shell at the same time [28]. In this situation, the integration contour can be pinched by two or more of poles of the propagators, and the scattering amplitude can be singular. The important feature of nonlocal theories is that the singularities of the local and nonlocal amplitudes are the same, and this means that the nonlocality does not affect the singularity structure of the scattering amplitudes.

我们注意到，该理论的非局域性仅编码在项 $B(k_i, p_h)$ 中，它保证了散射振幅中积分的紫外收敛性。根据假设，非局域形状因子是整函数，在有限复平面上没有零点，因此 $B(k_i, p_h)$ 在有限动量 k_i 和 p_h 下也没有零点或极点。因此，该函数不会在 (51) 的被积函数中引入传播子本身极点以外的其他极点，而传播子的极点与标量局域理论中的完全相同。事实上，和局域理论一样，(51) 的奇点通过求解朗道方程得到，朗道方程简单表达了两个或多个内线的四动量同时在壳的条件 [28]。在这种情况下，积分围道会被传播子的两个或多个极点夹住，散射振幅就会出现奇点。非局域理论的重要特点是，局域振幅和非局域振幅的奇点完全相同，这意味着非局域性不会改变散射振幅的奇点结构。

At this point, we are ready to give a prescription to calculate the imaginary part of the scattering amplitudes in nonlocal theory. For a generic amplitude (53), its imaginary part

至此，我们已经可以给出非局域理论中计算散射振幅虚部的规则了。对于一般形式的振幅 (53)，其虚部

$$\mathcal{M}(p_h) - \mathcal{M}^*(p_h) = \lim_{\varepsilon \to 0} \{\mathcal{M}(p_h, \varepsilon) - \mathcal{M}^*(p_h, \varepsilon)\} \tag{54}$$

evaluated at a specific singularity corresponding to the condition that the 4-momenta of certain internal lines are on-shell is given by replacing each on-shell propagator by

在对应特定内线四动量在壳条件的奇点处的取值，可通过将每个在壳传播子替换为

$$\frac{ie^{-H(\sigma p^2)}}{p^2 - m^2 + i\varepsilon} \to (-2\pi i)\,\sigma\left(p^0\right)\delta\left(p^2 - m^2 + i\varepsilon\right), \tag{55}$$

and replacing the integration region $\mathcal{I} \times \mathbb{R}^3$ of the loop momenta contained in such propagators with $\mathbb{R}^4$
.

再将这些传播子包含的圈动量积分区域 $\mathcal{I} \times \mathbb{R}^3$ 替换为 $\mathbb{R}^4$ 得到。

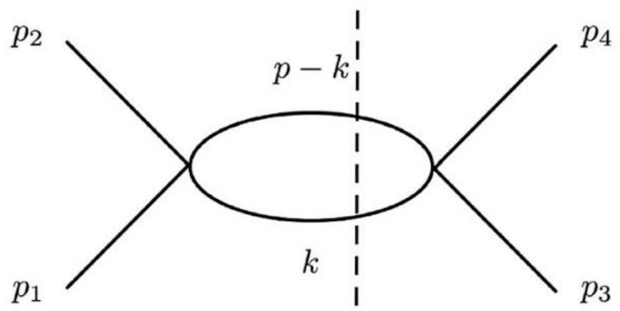
The expression obtained in this way is usually referred to as a cut diagram, since it is graphically represented by the same diagram as $\mathcal{M}$ in which the lines corresponding to the on-shell propagators are cut. For instance, for the amplitude (23), corresponding to the diagram in Fig. 1, the branch-cut singularity occurs at $E \geq 2m$, when the two internal lines can be on-shell. The imaginary part of this amplitude is given by (48) that is obtained by application of the Cutkosky rules to (23). The cut diagram corresponding to (48) is plotted in Fig. 4.

通过这种方式得到的表达式通常称为割图，因为它的图形表示就是在 $\mathcal{M}$ 的原图中切割对应在壳传播子的线得到的。例如，对于图 1 中对应图 (23) 的振幅，分支切割奇点出现在 $E \geq 2m$ 处，此时两个内线可以同时在壳。该振幅的虚部由 (48) 给出，(48) 就是对 (23) 应用卡特斯基规则得到的，对应的割图绘制在图 4 中。

If a solution of the Landau equations corresponding to a certain number of on-shell internal lines is such that, cutting all such on-shell lines, the diagram is divided into two parts, we say that the diagram has a branch-cut singularity corresponding to a normal threshold. This is because this singularity corresponds to the threshold of production of a real intermediate state, according to (14). If, on the contrary, cutting the on-shell propagators, the resulting diagram is not divided into two, we say that the amplitude has an anomalous threshold. For instance, in the triangle diagram in Fig. 5, the three internal lines can go on-shell at the same time. In this situation, cutting the three internal lines, the diagram is divided into three parts. In section ”Generic Amplitudes” we will show that, just as in the local theory, only the normal thresholds contribute to the imaginary part of the complex amplitudes, and this fact is fundamental for the unitarity of the theory.

> 如果朗道方程对应一定数量在壳内线的解满足: 切割所有这些在壳线后，图被分为两个部分，我们就称该图具有对应正规阈的分支切割奇点。这是因为根据 (14) 式，该奇点对应实中间态产生的阈。反之，如果切割所有在壳传播子后，所得的图没有被分成两部分，我们就称振幅具有反常阈。例如，在图 5 的三角图中，三条内线可以同时在壳。这种情况下，切割三条内线后图被分成三个部分。在“一般振幅”一节中我们将证明，和局域理论一样，只有正规阈对复振幅的虚部有贡献，这一性质是理论幺正性的基础。

Fig. 4 The cut diagram corresponding to (48) that gives the imaginary part of the one-loop contribution to the four-particle scattering amplitude in $\lambda\phi^4/4!$ theory



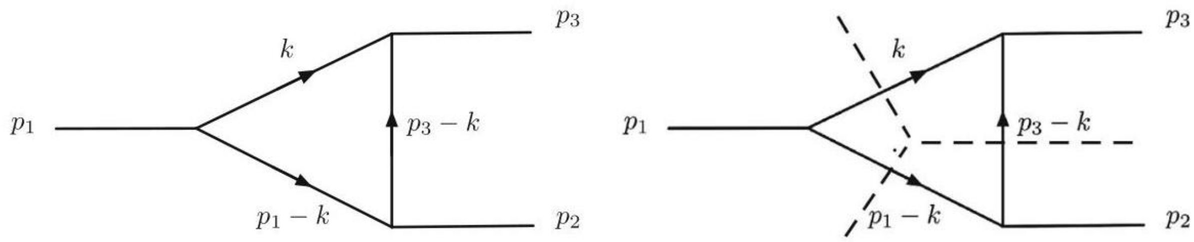


Fig. 5 On the left we plot the triangle diagram corresponding to (63). When the masses of the particles in the loop are chosen properly, the kinematics allows for the three internal lines to go on-shell at the same time. The corresponding cut diagram is plotted on the right. It consists of three vertex subdiagrams, so that it represents an anomalous threshold, and it has no correspondence in the sum in (14)

> 图 5 左侧为对应 (63) 的三角图。当恰当选择圈中粒子的质量后，运动学允许三条内线同时在壳。对应的割图绘制在右侧，它由三个顶点子图组成，因此代表一个反常阈，和 (14) 中的求和没有对应关系

Let us clarify an aspect of the Cutkosky rules. We note that the 4-vector p in the propagator is determined up to a sign, and this creates an ambiguity in the choice of the sign of p^0 in the $\sigma\left(p^0\right)$ in (46). Since $\sigma\left(p^0\right)$ selects the $p^0>0$ root of the delta functions, the right choice is the one corresponding to the positive energy of the intermediate real states. For instance, for the cut line of momenta $p-k$ in the diagram in Fig. 4, the right choice is

我们来澄清卡特斯基规则的一个方面。我们注意到传播子中的四矢量 p 仅在符号差一个正负号，这导致 (46) 式的 $\sigma\left(p^0\right)$ 中 p^0 的符号选择存在歧义。由于 $\sigma\left(p^0\right)$ 选择的是 δ 函数的 $p^0 > 0$ 根，正确选择对应中间实态的正能量。例如，对于图 4 中动量为 $p-k$ 的切割线，正确选择是

$$\frac{ie^{-H\left((p-k)^2\right)}}{(p-k)^2 - m^2 + i\varepsilon} \to (-2\pi i)\,\sigma\left(p^0 - k^0\right)\delta\left((p-k)^2 - m^2 + i\varepsilon\right)$$

(56)

so that the energy of the intermediate real state of momentum $p-k$ is positive, while the replacement

因此动量为 $p-k$ 的中间实态能量为正，而替换规则

$$\frac{ie^{-H\left((p-k)^2\right)}}{(p-k)^2 - m^2 + i\varepsilon}$$

$$= \frac{ie^{-H\left((k-p)^2\right)}}{(k-p)^2 - m^2 + i\varepsilon} \;\to\; (-2\pi i)\,\sigma\left(k^0 - p^0\right)\delta\left((k-p)^2 - m^2 + i\varepsilon\right)$$

(57)

would give a wrong result.

会给出错误结果。

It can be shown that if the Cutkosky rules give the imaginary part of any scattering amplitude $\mathcal{M}$ at certain singularity, and only normal thresholds contribute to $\mathcal{M} - \mathcal{M}^*$, then the condition (14) is always verified and the theory is unitary. This will be proven in section ”Generic Amplitudes”. Here, as an elucidatory example, we show that this statement is true for the amplitude (23), which has only a normal threshold at $E \geq 2m$. We have already proved that the imaginary part of (23) is given by the Cutkosky rules, that is, (48). Let us show that this expression implies (14). We introduce a fictitious integration variable $h = p - k$ by introducing a $\delta^{(4)}\left(p - h - k\right)$, so that

可以证明: 如果卡特科斯 ky 规则在特定奇点处给出任意散射振幅 $\mathcal{M}$ 的虚部, 且仅正常阈对 $\mathcal{M} - \mathcal{M}^*$ 有贡献，那么条件 (14) 总能成立，该理论是幺正的。这将在“一般振幅”一节中证明。本文在此给出一个说明性例子，证明该结论对振幅 (23) 成立，该振幅在 $E \geq 2m$ 处仅存在一个正常阈。我们已经证明 (23) 的虚部满足卡特科斯 ky 规则，即式 (48)。下面我们证明该表达式满足式 (14)。我们通过引入一个 $\delta^{(4)}\left(p-h-k\right)$ 引入虚构积分变量 $h = p - k$ ，因此

$$\mathcal{M}\left(p_h\right) - \mathcal{M}(p_h)^* = -\frac{\lambda^2}{2}\int_{\left(\mathbb{R}^4\times\mathbb{R}^4\right)} \frac{id^4k}{(2\pi)^4}\frac{d^4h}{(2\pi)^4}(-2\pi i)^2\sigma\left(h^0\right)\delta\left(h^2 - m^2\right)$$

$$\times\sigma\left(k^0\right)\delta\left(k^2 - m^2\right)(2\pi)^4\delta^{(4)}\left(p - h - k\right). \tag{58}$$

Then, we integrate the k^0 and h^0 variables, by obtaining

随后，我们对 k^0 和 h^0 变量积分，得到

$$\mathcal{M}(p_h) - \mathcal{M}(p_h)^* = \frac{i}{2}\int_{(\mathbb{R}^3\times\mathbb{R}^3)} \frac{d^3k}{2E_k(2\pi)^3}\frac{d^3h}{2E_h(2\pi)^3}\lambda^2(2\pi)^4\delta^{(4)}(p-h-k).$$

(59)

Finally, we note that for the diagram under consideration, the initial, final, and intermediate states $|a\rangle, | b\rangle$, and $|n\rangle$ are two boson states, and one has

最后我们注意到，对所考虑的图，初态、末态和中间态 $|a\rangle, | b\rangle$ 与 $|n\rangle$ 均为双玻色子态，因此有

$$\mathcal{M} = \mathcal{M}_{na} = \mathcal{M}_{nb} = \mathcal{M}_{ba} = \lambda + O\left(\lambda^2\right), \tag{60}$$

so that (59) gives

因此式 (59) 给出

$$\mathcal{M}_{ab}(p_h) - \mathcal{M}_{ba}(p_h)^*$$

$$= \frac{i}{2}\int_{(\mathbb{R}^3\times\mathbb{R}^3)} \frac{d^3k}{2E_k(2\pi)^3}\frac{d^3h}{2E_h(2\pi)^3}\mathcal{M}_{nb}^*\mathcal{M}_{na}(2\pi)^4\delta^{(4)}(p-h-k) + O\left(\lambda^3\right),$$

(61)

and (14) is verified at the leading λ^2 order. Diagrammatically, we see that $\mathcal{M}_{na}$ and $\mathcal{M}_{nb}^*$ correspond to the left and right vertex diagrams in the cut diagram in Fig. 4. This is a general feature of cut diagrams, so that the two parts of a cut diagram corresponding to a normal threshold represent the amplitudes of production of the intermediate states in (14).

且式 (14) 在领头 λ^2 阶下成立。从图上看，$\mathcal{M}_{na}$ 和 $\mathcal{M}_{nb}^*$ 分别对应图 4 中割图的左右顶点图。这是割图的普遍性质: 对应正常阈的割图的两个部分，就是式 (14) 中间态的产生振幅。

Before ending this section, we discuss briefly the Lorentz invariance of the amplitude (53). Let us consider an infinitesimal Lorentz transformation Λ . Applying this transformation to external and internal momenta, we have

在本节结束前，我们简要讨论振幅 (53) 的洛伦兹不变性。我们考虑一个无穷小洛伦兹变换 Λ 。将该变换应用于外动量和内动量，可得

$$\mathcal{M}' = -\frac{\lambda^V}{S_\#}\int_{(\mathcal{C}_L'\times\mathbb{R}^3)}\int_{(\mathcal{C}_{L-1}'\times\mathbb{R}^3)}\cdots\int_{(\mathcal{C}_1'\times\mathbb{R}^3)}\prod_{i=1}^{L}\frac{id^4k_i}{(2\pi)^4}\frac{1}{k_i^2-m^2+i\varepsilon}$$
$$\times\prod_{j=1}^{I-L}\frac{1}{q_j^2-m^2+i\varepsilon}B(k_i,p_h), \tag{62}$$

where we have used the fact that $B(k_i, p_h)$ depends on k_i and p_h through the square of their linear combinations, i.e., k_i^2, p_h^2 , and q_i^2 . Therefore, $B(k_i, p_h)$ is Lorentz invariant. Moreover, the contours $\mathcal{C}_i'$ are obtained from $\mathcal{C}_i$ through the Lorentz transformation Λ . Since Λ is infinitesimal, the two contours $\mathcal{C}_i'$ and $\mathcal{C}$ will be infinitesimally close, and we can safely assume that there will be no poles among them. Indeed, since the integrand in (62) is analytic, we can replace the integration contours $\mathcal{C}_i'$ with $\mathcal{C}$, obtaining $\mathcal{M}' = \mathcal{M}$ for an infinitesimal Lorentz transformation. Therefore, being invariant under infinitesimal Lorentz transformations, the amplitude (53) will be invariant under finite Lorentz transformations.

其中我们利用了以下事实: $B(k_i, p_h)$ 依赖于 k_i 和 p_h 通过其线性组合的平方，即 k_i^2, p_h^2 与 q_i^2 。因此，$B(k_i, p_h)$ 是洛伦兹不变的。此外，围道 $\mathcal{C}_i'$ 可通过洛伦兹变换 Λ 从 $\mathcal{C}_i$ 得到。由于 Λ 是无穷小变换，围道 $\mathcal{C}_i'$ 与 $\mathcal{C}$ 无穷接近，我们可以放心地认为它们之间没有极点。实际上，由于 (62) 中的被积函数是解析的，我们可以将积分围道 $\mathcal{C}_i'$ 替换为 $\mathcal{C}$ ，对无穷小洛伦兹变换得到 $\mathcal{M}' = \mathcal{M}$ 。因此，既然振幅 (53) 在无穷小洛伦兹变换下不变，它在有限洛伦兹变换下也不变。

One-Loop Amplitude with an Anomalous Threshold

具有反常阈值的单圈振幅

We have already claimed that the imaginary part of $\mathcal{M}$ at a specific singularity is given by the Cutkosky rules, postponing the proof of this statement to section ”Generic Amplitudes”. The expression obtained in this way is called cut diagram, since it is graphically represented by the same diagram as $\mathcal{M}$ in which the lines corresponding to the on-shell propagators are cut, see Figs. 4 and 7. As we mentioned before, the singularities corresponding to cuts that divide the original diagram into two are usually referred as normal thresholds. For such normal thresholds it is easy to see that (14) is a straightforward consequence of the Cutkosky rules, as we did in the previous section for the one-loop amplitude (23). However, in some cases, cutting the on-shell propagators, the cut diagram is not divided into two, and the imaginary part of the amplitude cannot be recast as in (14). The corresponding solutions of the Landau equations are usually referred as anomalous thresholds.

我们已经指出，$\mathcal{M}$ 在特定奇点处的虚部由 Cutkosky 规则给出，该结论的证明我们放到“一般振幅”章节中讨论。通过这种方式得到的表达式称为切割图，因为它的图形表示与 $\mathcal{M}$ 的原图形相同，只是对应于在壳传播子的线被切割，参见图 4 和图 7。如前文所述，将原 diagram 分割为两部分的切割对应的奇点通常称为正常阈值。对于这类正常阈值，不难看出式 (14) 是 Cutkosky 规则的直接推论，正如我们在前一章节对单圈振幅 (23) 所做的推导。但在某些情况下，切割在壳传播子后，切割图并未被分割为两部分，因此振幅的虚部无法写成式 (14) 的形式。朗道方程的这类对应解通常称为反常阈值。

Therefore, to prove the unitarity of the nonlocal scalar theory, one has to show that only the normal thresholds contribute to the imaginary part of $\mathcal{M}$. This can be achieved by showing that the diagrams that contribute to (14) in the nonlocal case are all and only those that contribute in the case of the local scalar field theory. Indeed, there is no contribution from anomalous thresholds, because we know that anomalous thresholds do not contribute in the local case, as the local theory is unitary. This argument will be used in section ”Generic Amplitudes” to prove that the unitarity condition (14) is fulfilled by generic amplitudes.

因此，要证明非局域标量理论的幺正性，必须证明只有正常阈值对 $\mathcal{M}$ 的虚部有贡献。我们可以通过证明: 非局域情形下对式 (14) 有贡献的 diagram 与局域标量场论情形下的贡献 diagram 完全一致，即可得到上述结论。事实上，反常阈值没有贡献，因为我们知道局域理论是幺正的，反常阈值在局域情形下本就没有贡献。我们将在“一般振幅”章节中使用这一论证，证明一般振幅都满足幺正条件 (14)。

Here, as an illustrative example, we consider the triangle diagram in Fig. 5, in a theory with three interacting scalar fields with masses m_1, m_2 , and m_3 respectively, and cubic interactions. When these masses are chosen properly, the kinematics allows for the three internal lines to go on-shell at the same time, for some values of the external momenta. This situation corresponds to an anomalous threshold since, cutting the lines of the three on-shell propagators, the diagram is divided into three parts, see Fig. 4. However, $\mathcal{M} - \mathcal{M}^*$ contains only products of two amplitudes; indeed the cut diagram in the right-hand side of Fig. 4 has no correspondence in (14). An explicit calculation shows that $\mathcal{M} - \mathcal{M}^*$ contains only contributions from normal thresholds with two internal lines on-shell. These are the same cut diagrams that contribute in the case of the corresponding local theory obtained by setting $H(z) = 0$, as expected. This is due to the fact that $B(k, p_1, p_2, p_3)$ has no zeros or poles, and thus the local and nonlocal amplitudes have the same singularity structure.

此处我们以图 5 的三角图作为说明性示例，该图取自存在三个质量分别为 m_1, m_2 和 m_3 的相互作用标量场、且具有三次相互作用的理论中。当质量选取合适时，对某些外动量取值，运动学允许三条内线同时处于在壳态。这种情况对应一个反常阈值，因为切割三条在壳传播子的线后，图被分割为三个部分，参见图 4。但 $\mathcal{M} - \mathcal{M}^*$ 仅包含两个振幅的乘积；事实上，图 4 右侧的切割图在式 (14) 中没有对应项。显式计算表明，$\mathcal{M} - \mathcal{M}^*$ 仅包含两条内线在壳的正常阈值贡献，正如预期，这些贡献与令 $H(z) = 0$ 后得到的对应局域理论中的贡献切割图完全一致。这是因为 $B(k, p_1, p_2, p_3)$ 没有零点也没有极点，因此局域和非局域振幅具有相同的奇点结构。

Let us assume that p_1 is the momentum of the particle corresponding to the initial state, and p_2 and p_3 are the momenta of the particles in the final state of the triangle diagram in Fig. 5. As usual, energy conservation implies $p_1 = p_2 + p_3$. The amplitude of the triangle diagram is

设 p_1 为三角图初态对应粒子的动量，p_2 和 p_3 为三角图末态粒子的动量，见图 5。和通常一样，能量守恒给出 $p_1 = p_2 + p_3$ 。三角图的振幅为

$$\mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^2}{2} \int_{(\mathcal{C}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(k-p_1)^2 - m_2^2 + i\varepsilon} \frac{B(k, p_1, p_2, p_3)}{(k-p_3)^2 - m_3^2 + i\varepsilon}, \tag{63}$$

where $B(k, p_1, p_2, p_3) \equiv \mathcal{V}(p_1, k, p-k)\, \mathcal{V}(p_1 - k, p_2, p_3 - k)\, \mathcal{V}(p_3 - k, p_3, k)$, and $\mathcal{V}(v_1, v_2, v_3) = e^{-\frac{1}{2}\sum_{i=1}^{3} H\left(\sigma(v_i)^2\right)}$. Since we want this amplitude to be nonzero, we assume $p_1^2 = M^2 > p_2^2 + p_3^2$. We also assume that the masses of the internal propagators are different, so that the three propagators can go on-shell together. For instance, this happens in the case $m_1 = m_2 = m$, and $p_2^2 = p_3^2 = p^2 > m^2 + m_3^2$, when one has an anomalous threshold at

其中 $B(k, p_1, p_2, p_3) \equiv \mathcal{V}(p_1, k, p-k)\, \mathcal{V}(p_1 - k, p_2, p_3 - k)\, \mathcal{V}(p_3 - k, p_3, k)$ ，且 $\mathcal{V}(v_1, v_2, v_3) = e^{-\frac{1}{2}\sum_{i=1}^{3} H\left(\sigma(v_i)^2\right)}$。由于我们要求该振幅非零，故假设 $p_1^2 = M^2 > p_2^2 + p_3^2$。我们还假设内传播子质量各不相同，因此三个传播子可以同时在壳。例如，当存在反常阈值时，这种情况发生在 $m_1 = m_2 = m$ 且 $p_2^2 = p_3^2 = p^2 > m^2 + m_3^2$ ，即

$$p_1^2 = 4m^2 - \frac{\left(p^2 - m^2 - m_3^2\right)^2}{m_3^2}, \tag{64}$$

see [28] for details. Anomalous thresholds occur also in other configurations, e.g., in the case $m_1 = m_3 = m, m_2 = 2m, p_2^2 = m^2, p_3^2 = 4m^2$ for $p_1^2 = 9m^2$.

详见文献 [28]。反常阈值也会出现在其他组态中，例如，对于 $p_1^2 = 9m^2$ ，在 $m_1 = m_3 = m, m_2 = 2m, p_2^2 = m^2, p_3^2 = 4m^2$ 的情况下就会出现。

The procedure of analytic continuation prescribes that the integration contour $\mathcal{C}$ is obtained by deforming the imaginary axis of the complex k^0 plane around those poles of the propagators in (63) that pass through $\mathcal{I}$ when the energies p_1^0, p_2^0, p_3^0 become real and positive. The poles of the first propagator are

解析延拓过程规定，积分围道 $\mathcal{C}$ 是通过对复 k^0 平面的虚轴做变形得到的，变形绕过 (63) 中那些当能量 p_1^0, p_2^0, p_3^0 取实正值时穿过 $\mathcal{I}$ 的传播子极点。第一个传播子的极点为

$$\bar{k}_{1,2}^0 = \pm\sqrt{\vec{k}^2 + m_1^2 - i\varepsilon}. \tag{65}$$

Such poles do not depend on the external energies and remain always far from the imaginary axis. Indeed, $\bar{k}_{1,2}^0$ do not pinch $\mathcal{I}$. The poles of the second and third propagators are

这类极点不依赖于外能，始终远离虚轴。事实上，$\bar{k}_{1,2}^0$ 不会对 $\mathcal{I}$ 形成夹挤。第二个和第三个传播子的极点为

$$\bar{k}_{3,4}^0 = p_1^0 \pm \sqrt{\left(\vec{k} - \vec{p}_1\right)^2 + m_2^2 - i\varepsilon},\ \bar{k}_{5,6}^0 = p_3^0 \pm \sqrt{\left(\vec{k} - \vec{p}_3\right)^2 + m_3^2 - i\varepsilon}.$$

(66)

When the external energies are purely imaginary, $\bar{k}_3^0$ and $\bar{k}_5^0$ are at the right and $\bar{k}_4^0$ and $\bar{k}_6^0$ are at the left of the imaginary axis of the k^0 plane. When the external energies are moved to their physical real and positive values, such poles move to the right, and it happens that the poles $\bar{k}_4^0$ and $\bar{k}_6^0$ pass through the imaginary axis for some values of the loop momenta k . Therefore, the integration contour $\mathcal{C}$ is obtained by deforming the imaginary axis $\mathcal{I}$ around the poles $\bar{k}_4^0$ and $\bar{k}_6^0$ when $\Re\{\bar{k}_4^0\} > 0$ and $\Re\{\bar{k}_6^0\} > 0$. This picture is represented in Fig. 6. From the same figure we see that the integral along the deformed contour $\mathcal{C}$ can be broken into two parts: The first corresponding to the integration along $\mathcal{I}$, and the second given by the contribution of the residues at the poles $\bar{k}_4^0$ and $\bar{k}_6^0$. Therefore, we can write

当外能为纯虚数时，$\bar{k}_3^0$ 和 $\bar{k}_5^0$ 位于 k^0 平面虚轴的右侧，$\bar{k}_4^0$ 和 $\bar{k}_6^0$ 位于虚轴左侧。当外能移动到物理实正值区域时，这些极点向右移动，对于某些圈动量 k 的值，极点 $\bar{k}_4^0$ 和 $\bar{k}_6^0$ 会穿过虚轴。因此，当 $\Re\{\bar{k}_4^0\} > 0$ 和 $\Re\{\bar{k}_6^0\} > 0$ 时，积分围道 $\mathcal{C}$ 是通过将虚轴 $\mathcal{I}$ 变形绕过极点 $\bar{k}_4^0$ 和 $\bar{k}_6^0$ 得到的。该图像如图 6 所示。从图中可以看出，变形围道 $\mathcal{C}$ 上的积分可以拆分为两部分: 第一部分是沿 $\mathcal{I}$ 的积分，第二部分是极点 $\bar{k}_4^0$ 和 $\bar{k}_6^0$ 处的留数贡献。因此我们可以写出

$$\mathcal{M}(p_h, \varepsilon) = -\frac{\lambda^2}{2}\left[\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{1}{k^2 - m_1^2 + i\varepsilon} \frac{B(k, p_1, p_2, p_3)}{(k - p_1)^2 - m_2^2 + i\varepsilon} \frac{1}{(k - p_3)^2 - m_3^2 + i\varepsilon}\right.$$

$$+\int_{(\mathbb{R}^3)} \frac{id^3k}{(2\pi)^4} 2\pi i\sigma\left(\Re\{\bar{k}_4^0\}\right) \operatorname{Res}\left\{\frac{B(k, p_1, p_2, p_3)}{k^2 - m_1^2 + i\varepsilon} \frac{1}{(k - p_1)^2 - m_2^2 + i\varepsilon} \frac{1}{(k - p_3)^2 - m_3^2 + i\varepsilon}, \bar{k}_4^0\right\}$$

$$+\int_{(\mathbb{R}^3)} \frac{id^3k}{(2\pi)^4} 2\pi i\sigma\left(\Re\left\{\bar{k}_6^0\right\}\right) \mathrm{Res}\left\{\frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon}\frac{1}{(k-p_1)^2-m_2^2+i\varepsilon}\frac{1}{(k-p_3)^2-m_3^2+i\varepsilon}, \bar{k}_6^0\right\}.$$

(67)

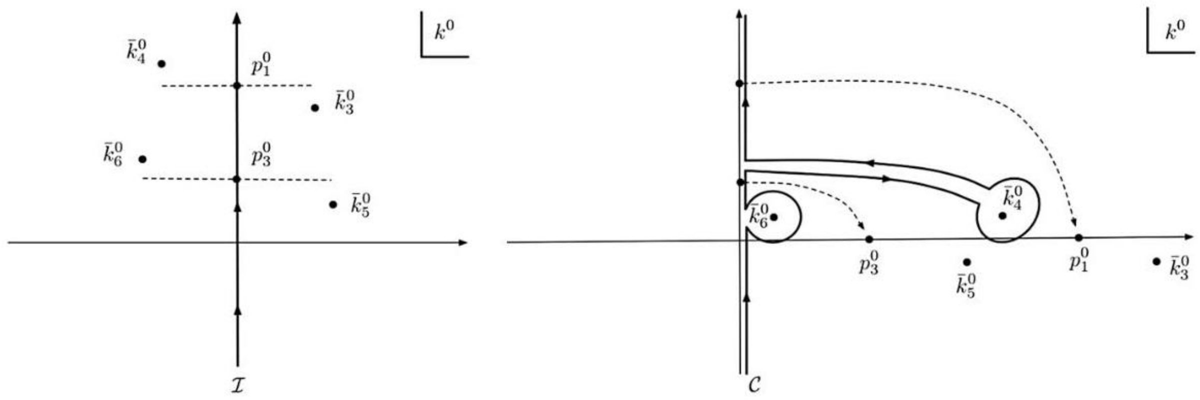


Fig. 6 (Left) We plot the poles $\bar{k}_3^0, \bar{k}_4^0, \bar{k}_5^0, \bar{k}_6^0$ on the complex k^0 plane, when the external energies p_1^0 and p_3^0 are purely imaginary. (Right) We plot the same poles when the external energies are moved to their physical real and positive values. The poles $\bar{k}_4^0$ and $\bar{k}_6^0$ pass through the imaginary axis for some values of the loop momenta k . The integration contour $\mathcal{C}$ is obtained by deforming the imaginary axis $\mathcal{I}$ around $\bar{k}_4^0$ and $\bar{k}_6^0$

图 6(左) 我们绘制了外能 p_1^0 和 p_3^0 为纯虚数时，极点 $\bar{k}_3^0, \bar{k}_4^0, \bar{k}_5^0, \bar{k}_6^0$ 在复 k^0 平面上的分布。(右) 我们绘制了外能取物理实正值时相同极点的分布。对于某些圈动量 k 的取值，极点 $\bar{k}_4^0$ 和 $\bar{k}_6^0$ 会穿过虚轴。积分围道 $\mathcal{C}$ 是将虚轴 $\mathcal{I}$ 绕过 $\bar{k}_4^0$ 和 $\bar{k}_6^0$ 形变得到的

Evaluating the residues in (67), one has

计算 (67) 中的留数可得

$$\mathrm{Res}\left\{\frac{1}{k^2-m_1^2+i\varepsilon}\frac{1}{(k-p_1)^2-m_2^2+i\varepsilon}\frac{B(k,p_1,p_2,p_3)}{(k-p_3)^2-m_3^2+i\varepsilon}, \bar{k}_4^0\right\} =$$

$$\left(-\frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon}\frac{1}{2\sqrt{\left(\vec{k}-\vec{p}_1\right)^2+m_2^2-i\varepsilon}}\frac{1}{(k-p_3)^2-m_3^2+i\varepsilon}\right)\Bigg|_{k^0=\vec{k}_4^0},$$

(68)

and

且

$$\mathrm{Res}\left\{\frac{1}{k^2-m^2+i\varepsilon}\frac{1}{(k-p_1)^2-m^2+i\varepsilon}\frac{B(k,p_1,p_2,p_3)}{(k-p_3)^2-m^2+i\varepsilon}, \bar{k}_6^0\right\}$$

$$= \left.\left(-\frac{B(k,p_1,p_2,p_3)}{k^2-m^2+i\varepsilon} \frac{1}{2\sqrt{\left(\vec{k}-\vec{p}_3\right)^2+m^2-i\varepsilon}} \frac{1}{(k-p_1)^2-m^2+i\varepsilon} \right)\right|_{k^0=\bar{k}_6^0},$$

(69)

so that the amplitude (67) becomes

因此振幅 (67) 可写为

$$\mathcal{M}(p_h,\varepsilon) = -\frac{\lambda^2}{2}\left[\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{1}{k^2-m_1^2+i\varepsilon} \frac{B(k,p_1,p_2,p_3)}{(k-p_1)^2-m_2^2+i\varepsilon} \frac{1}{(k-p_3)^2-m_3^2+i\varepsilon}\right.$$
$$+\int_{(\mathbb{R}^3)} \frac{id^3k}{(2\pi)^4} (-2\pi i)\,\sigma\left(\Re\left\{\bar{k}_4^0\right\}\right)$$
$$\times\left.\left(\frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon} \frac{1}{2\sqrt{\left(\vec{k}-\vec{p}_1\right)^2+m_2^2-i\varepsilon}} \frac{1}{(k-p_3)^2-m_3^2+i\varepsilon} \right)\right|_{k^0=\bar{k}_4^0}$$
$$+\int_{(\mathbb{R}^3)} \frac{id^3k}{(2\pi)^4} (-2\pi i)\,\sigma\left(\Re\left\{\bar{k}_6^0\right\}\right)$$
$$\times\left.\left(-\frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon} \frac{1}{2\sqrt{\left(\vec{k}-\vec{p}_3\right)^2+m_3^2-i\varepsilon}} \frac{1}{(k-p_1)^2-m_2^2+i\varepsilon} \right)\right|_{k^0=\bar{k}_6^0}.$$

(70)

Since $\left(\vec{k}-\vec{p}\right)^2+m^2>0$, we can omit the $i\varepsilon$ term in the square roots in (70), so that it can be recast by means of (36) as

由于 $\left(\vec{k}-\vec{p}\right)^2+m^2>0$ ，我们可以略去 (70) 中平方根里的 $i\varepsilon$ 项，再通过 (36) 将其改写为

$$\mathcal{M}(p_h,\varepsilon) = -\frac{\lambda^2}{2}\left[\int_{(\mathcal{I}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{1}{k^2-m_1^2+i\varepsilon} \frac{B(k,p_1,p_2,p_3)}{(k-p_1)^2-m_2^2+i\varepsilon} \frac{1}{(k-p_3)^2-m_3^2+i\varepsilon}\right.$$
$$+\int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} (-2\pi i)\,\sigma\left(k^0\right)\sigma\left(p_1^0-k^0\right) \frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon} \frac{\delta\left((k-p_1)^2-m_2^2\right)}{(k-p_3)^2-m_3^2+i\varepsilon}$$
$$+\int_{(\mathbb{R}^4)} \frac{id^4k}{(2\pi)^4} (-2\pi i)\,\sigma\left(k^0\right)\sigma\left(p_3^0-k^0\right) \frac{B(k,p_1,p_2,p_3)}{k^2-m_1^2+i\varepsilon} \frac{\delta\left((k-p_3)^2-m^2\right)}{(k-p_1)^2-m_2^2+i\varepsilon}.$$

(71)

Following the same argument used in the case of (38), one can show that the first integral in (71) is real in the limit $\varepsilon\to 0$. In fact, $k\in\mathcal{I}\times\mathbb{R}^3$ while $p_i\in\mathbb{R}^4$; indeed the three denominators in the integrand are never zero, except that on a subset of the integration domains with null measure, so that they become real when $\varepsilon\to 0$. The proof is completed by writing the complex conjugate of this integral and redefining the integration variable as $k^0\to -k^0$.

沿用 (38) 情形的推导思路, 可以证明 (71) 中的第一个积分在极限 $\varepsilon \to 0$ 下为实数。实际上, $k \in \mathcal{I} \times \mathbb{R}^3$ 而 $p_i \in \mathbb{R}^4$；不难看出被积函数中的三个分母仅在零测度积分子集上为零，其余位置均不为零，因此当 $\varepsilon \to 0$ 时该积分变为实数。对该积分取复共轭并将积分变量重新定义为 $k^0 \to -k^0$ 即可完成证明。

Therefore, the imaginary part of (71) comes only from the last two integrals. Using (40), one finds

因此，(71) 的虚部仅来自最后两个积分。利用 (40) 可得

$$\begin{aligned}
&\mathcal{M}(p_h,\varepsilon)-\mathcal{M}(p_h,\varepsilon)^* \\
&= -\frac{\lambda^2}{2}\int_{(\mathbb{R}^4)}\frac{id^4k}{(2\pi)^4}(-2\pi i)^2\Bigg[\sigma(k^0)\,\sigma(p_1^0-k^0)\Bigg(\frac{\delta(k^2-m_1^2)}{(k-p_3)^2-m_3^2+i\varepsilon} \\
&\quad+\frac{\delta\left((k-p_3)^2-m_3^2\right)}{k^2-m_1^2+i\varepsilon}\Bigg)\delta\left((p_1-k)^2-m_2^2\right) \\
&\quad+\sigma(k^0)\,\sigma(p_3^0-k^0)\Bigg(\frac{\delta(k^2-m_1^2)}{(k-p_1)^2-m_2^2+i\varepsilon}+\frac{\delta\left((k-p_1)^2-m_2^2\right)}{k^2-m_1^2+i\varepsilon}\Bigg) \\
&\quad\times\delta\left((p_3-k)^2-m_3^2\right)\Bigg]B(k,p_1,p_2,p_3)\,.
\end{aligned} \tag{72}$$

Finally, we note that the kinematics implies that, since $p_1^2 > p_2^2+p_3^2$, the only propagators that can go on-shell together are those involving the momenta k and p_1-k , so that one has

最后我们注意到，运动学条件表明，由于 $p_1^2 > p_2^2+p_3^2$，只有同时包含动量 k 和 p_1-k 的传播子可以同时在壳，因此有

$$\begin{aligned}
\mathcal{M}(p_h,\varepsilon)-\mathcal{M}(p_h,\varepsilon)^* &= -\frac{\lambda^2}{2}\int_{(\mathbb{R}^4)}\frac{id^4k}{(2\pi)^4}(-2\pi i)^2\frac{e^{-H\left(\sigma^2(k-p_3)^2\right)}}{(k-p_3)^2-m_3^2+i\varepsilon} \\
&\times\sigma(k^0)\,\sigma(p_1^0-k^0)\,\delta(k^2-m_1^2)\,\delta\left((p_1-k)^2-m_2^2\right),
\end{aligned}$$

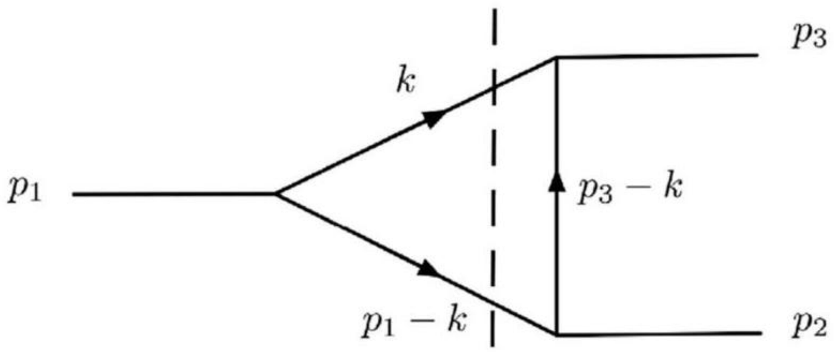
(73)

where we have replaced $B(k,p_1,p_2,p_3)=e^{-H\left(\sigma^2(k-p_3)^2\right)}$, since the momenta k and $k-p$ are on-shell.

其中由于动量 k 和 $k-p$ 在壳，我们替换了 $B(k,p_1,p_2,p_3)=e^{-H\left(\sigma^2(k-p_3)^2\right)}$

Fig. 7 We plot the only cut diagram that contributes to $\mathcal{M}-\mathcal{M}^*$. In fact, due to the kinematic condition $p_1^2 > p_2^2+p_3^2$, the only couple of propagators that can go on-shell together is that involving the momenta k and p_1-k

图 7 我们绘制了对 $\mathcal{M}-\mathcal{M}^*$ 有贡献的唯一切割图。实际上，由于运动学条件 $p_1^2 > p_2^2+p_3^2$，只有包含动量 k 和 p_1-k 的一对传播子可以同时在壳



From (73) we see that, due to the kinematics, only one of the cut diagrams with two cut lines, more precisely, the one shown in Fig. 7, contributes to $\mathcal{M} - \mathcal{M}^*$, while the anomalous threshold of the triangle diagram does not contribute to $\mathcal{M} - \mathcal{M}^*$. Furthermore, we have verified explicitly that the imaginary part of the scattering amplitude at such normal threshold is given by the Cutkosky rules.

从 (73) 可以看出，受运动学限制，具有两条切割线的切割图中，仅有图 7 所示的这一个对 $\mathcal{M} - \mathcal{M}^*$ 有贡献，三角形图的反常阈值对 $\mathcal{M} - \mathcal{M}^*$ 没有贡献。此外我们明确验证了，该正常阈值处散射振幅的虚部满足卡特斯基规则。

Generic Amplitudes

广义振幅

In this section we show that the imaginary part of a generic scattering amplitude $\mathcal{M}$ at a specific singularity is given by the application of the Cutkosky rules. We also show that in the sum in (14) there is no contribution from anomalous thresholds. These two facts imply that the nonlocal theories (1) and (5) are unitary.

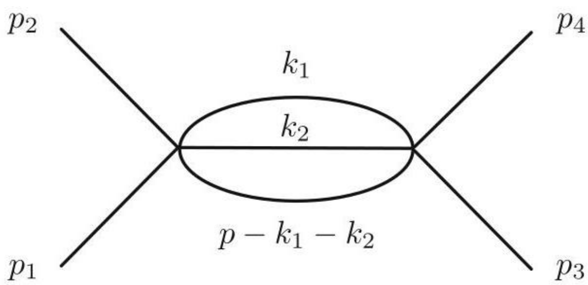
本节我们证明，特定奇点处广义散射振幅 $\mathcal{M}$ 的虚部可由卡特斯基规则给出。我们还证明 (14) 式的求和中不存在反常阈值贡献。这两点说明非局域理论 (1) 和 (5) 是幺正的。

We have already pointed out that the nonlocality does not affect the singularity structure of the scattering amplitudes. Indeed, the singularities of (53) are found by solving the Landau equations, which impose that two or more of poles of the propagators coincide for some value of the external and loop momenta, which implies that two or more internal momenta are on-shell at the same time. However, we have to emphasize that not all the solutions of the Landau equations correspond to a branch-cut singularity of $\mathcal{M}$, since only the poles that pinch one of the integration contours $\mathcal{C}_i$ give rise to a singularity. In fact, it might happen that the integration domain is not constrained between two overlapping poles, indeed these poles are still far from the integration contour, and the loop integrals are finite. This fact is intimately related to the absence of contributions from anomalous thresholds in (14). In fact, anomalous thresholds correspond to solutions of the Landau equations, which do not give rise to a branch-cut singularity of the amplitude. For instance, in the diagram in Fig. 8, the situation in which two of the three internal lines are on-shell simultaneously does not correspond to a branch-cut singularity, while the amplitude has a normal threshold when the three internal lines are on-shell at the same time. On the other side, in the diagram in Fig. 5, the three internal lines can go on-shell simultaneously for certain values of the external momenta, but this does not imply that the corresponding scattering amplitude has a branch-cut singularity. On the contrary, this diagram has a branch-cut singularity when the internal lines go on-shell in pairs.

我们已经指出，非局域性不改变散射振幅的奇点结构。事实上，(53) 的奇点可通过求解朗道方程得到: 朗道方程要求，对于给定的外线动量和圈动量取值，传播子的两个及以上极点重合，这意味着两个及以上内动量同时在壳。但必须强调，并非朗道方程的所有解都对应 $\mathcal{M}$ 的割线奇点，因为只有夹住其中一条积分围道 $\mathcal{C}_i$ 的极点才会产生奇点。实际上，积分域可能不会被两个重叠极点限制，若这些极点离积分围道仍然较远，圈积分就是有限的。这一性质与 (14) 式中不存在反常阈值贡献密切相关。反常阈值本身确实是朗道方程的解，但它不会给出振幅的割线奇点。例如，图 8 的图中，三条内线中的两条同时在壳并不对应割线奇点，而当三条内线同时在壳时，振幅存在正规阈值。另一方面，图 5 的图中，对于特定外动量取值，三条内线可以同时在壳，但这并不意味着对应的散射振幅存在割线奇点，相反，该图在内线成对在壳时才存在割线奇点。

Fig. 8 A two Feynman loops diagram with three internal lines

图 8 含三条内线的双费曼图



In order to prove the unitarity of the nonlocal theory, we have to evaluate the imaginary part of a generic nonlocal scattering amplitude, that is,

为了证明非局域理论的幺正性，我们需要计算广义非局域散射振幅的虚部，即

$$\mathcal{M}(p_h) - \mathcal{M}^*(p_h) \equiv \lim_{\varepsilon \to 0} \{\mathcal{M}(p_h, \varepsilon) - \mathcal{M}^*(p_h, \varepsilon)\}. \tag{74}$$

It can be shown [29] that

文献 [29] 已证明

$$\lim_{\varepsilon \to 0} \{\mathcal{M}(p_h, \varepsilon) - \mathcal{M}^*(p_h, \varepsilon)\} = \lim_{\varepsilon \to 0} \{\mathcal{M}(p_h, \varepsilon) - \mathcal{M}(p_h, -\varepsilon)\}, \tag{75}$$

which implies that the imaginary part of the amplitude at a specific threshold is given by the discontinuity of $\mathcal{M}$ at the corresponding branch-cut singularity. For instance, in the case of the one-loop amplitude (23), this is given by the jump in the complex amplitude at the branch-cut singularity at $E \geq 2m$, that is,

这说明特定阈值处振幅的虚部由对应割线奇点处 $\mathcal{M}$ 的间断给出。例如，对于单圈振幅 (23)，虚部由复振幅在 $E \geq 2m$ 处割线奇点的跳跃给出，即

$$\mathcal{M}(p_h) - \mathcal{M}^*(p_h) = \lim_{\varepsilon \to 0} \left\{ \mathcal{M}\left(E_h + i\varepsilon, \vec{p}_h\right) - \mathcal{M}\left(E_h - i\varepsilon, \vec{p}_h\right) \right\}. \tag{76}$$

Starting from this observation, Cutkosky has showed [34] that, in the case of local theories, the discontinuity of $\mathcal{M}$ at the branch cut is obtained by replacing in (53) each propagator that goes on-shell at the singularity, with a term $(-2\pi i)\,\delta\left(p^2 - m^2\right)$. This is basically the same result that we obtained in the case of the nonlocal one-loop amplitude (23), when we have separated the contribution of the residue in (34) and then obtained (48). Indeed, in section ”One-Loop Diagram”, we have already recovered the Cutkosky rules for (23).

基于这一观察，卡特斯基在文献 [34] 中证明，局域理论中割线处 $\mathcal{M}$ 的间断可通过将 (53) 中在奇点处成为在壳的每个传播子替换为项 $(-2\pi i)\,\delta\left(p^2 - m^2\right)$ 得到。这一结果与我们之前分析非局域单圈振幅 (23) 得到的结果基本一致——我们在分析中分离了 (34) 中留数的贡献，最终得到了 (48)。事实上，我们在“单圈图”一节已经推导出了 (23) 满足的卡特斯基规则。

Since the nonlocal and local scattering amplitudes have the same poles, one expects that the argument used by Cutkosky is not affected by the presence of the nonlocal form factors and that the Cutkosky rules are still valid for nonlocal theories. The only difference with respect to the local case is that one has to replace the integration region $\mathcal{C} \times \mathbb{R}^3$ with $\mathbb{R}^4$ for all the loop momenta contained in the on-shell propagators. We can show that these statements are correct, following the proof outlined in [34].

由于非局域和局域散射振幅的极点完全相同，可以预期卡特斯基的论证不受非局域形状因子的影响，卡特斯基规则对非局域理论仍然成立。与局域情形的唯一区别是，对于在壳传播子包含的所有圈动量，需要将积分区域 $\mathcal{C} \times \mathbb{R}^3$ 替换为 $\mathbb{R}^4$ 。我们可以遵循文献 [34] 给出的证明，验证上述结论成立。

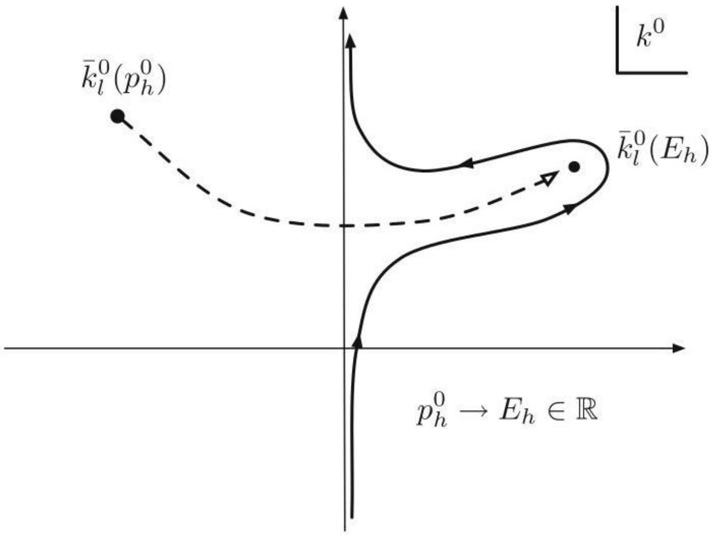
In order to proceed, we must determine how the integration contour $\left(\mathcal{I} \times \mathbb{R}^3\right)$ for the i -th loop variable k_i in (51) is deformed into $\left(\mathcal{C}_i \times \mathbb{R}^3\right)$. In what follows, we will give a prescription for finding $\left(\mathcal{C}_i \times \mathbb{R}^3\right)$ proceeding iteratively. Since we want to integrate (51) one loop at time, it is useful to define how the deformed integration contour is obtained for a generic integral in one-loop variable. Let us consider the following integral:

为了继续推导，我们必须确定式 (51) 中圈变量 k_i 的第 i 圈积分围道 $\left(\mathcal{I} \times \mathbb{R}^3\right)$ 应如何形变得到 $\left(\mathcal{C}_i \times \mathbb{R}^3\right)$ 。下文我们将给出一种通过迭代寻找 $\left(\mathcal{C}_i \times \mathbb{R}^3\right)$ 的方法。由于我们计划每次只对式 (51) 中的一个圈做积分，因此先定义单圈变量一般积分的形变积分围道构造方法会更方便。我们考虑如下积分:

$$M(p_h) = \int_{\mathcal{I} \times \mathbb{R}^3} f(k, p_h)\, dk^0 d^3k, \tag{77}$$

Fig. 9 The pole $\bar{k}_l^0\left(p_h^0\right)$ moves through the imaginary axes $\mathcal{I}$ when $p_h^0 \to E_h^0$, and the integration contour $\mathcal{C}$ is obtained by deforming $\mathcal{I}$ around $\bar{k}_l^0(E_h)$. Note that we have omitted the dependence of the pole $\bar{k}_l^0$ on $\vec{k}$ and $\vec{p}$

图 9 当 $p_h^0 \to E_h^0$ 时极点 $\bar{k}_l^0\left(p_h^0\right)$ 穿过虚轴 $\mathcal{I}$，变形围道 $\mathcal{C}$ 是将 $\mathcal{I}$ 绕过 $\bar{k}_l^0\left(E_h\right)$ 得到。注意我们此处省略了极点 $\bar{k}_l^0$ 对 $\vec{k}$ 和 $\vec{p}$ 的依赖关系



where p_h represents the external momenta, and the function $f\left(k, p_h\right)$ has m poles $\bar{k}_l^0\left(\vec{k}, p_h\right)$ with $l = 1, 2, \ldots m$ in the complex k^0 plane. We assume that the poles $\bar{k}_l^0\left(\vec{k}, p_h\right)$ are far from the integration contour when the external energies p_h^0 are purely imaginary, as in (51). Then, we continue analytically (77) to real and positive external energies. Indeed, when $p_h^0 \to E_h \in \mathbb{R}_0^+$, the poles $\bar{k}_l^0\left(\vec{k}, p_h\right)$ will move in the k^0 complex plane, and some of them will pass through the imaginary axis $\mathcal{I}$. Therefore, the analytic continuation of $M\left(p_h\right)$ is obtained from (77) by deforming the integration contour $\mathcal{I}$ around the poles that cross the imaginary axis, as schematically plotted in Fig. 9, so that one has

其中 p_h 代表外动量，函数 $f\left(k, p_h\right)$ 在复 k^0 平面上存在 m 个极点 $\bar{k}_l^0\left(\vec{k}, p_h\right)$，极点满足 $l = 1, 2, \ldots m$。我们假设，如式 (51) 的情况，当外能 p_h^0 为纯虚数时，极点 $\bar{k}_l^0\left(\vec{k}, p_h\right)$ 远离积分围道。随后我们将式 (77) 解析延拓到实正外能。实际上，当 $p_h^0 \to E_h \in \mathbb{R}_0^+$ 时，所有极点 $\bar{k}_l^0\left(\vec{k}, p_h\right)$ 会在 k^0 复平面内移动，部分极点会穿过虚轴 $\mathcal{I}$。因此，我们需要将积分围道 $\mathcal{I}$ 变形，绕过穿过虚轴的极点，得到 $M\left(p_h\right)$ 的解析延拓结果，该过程如图 9 示意所示，即有

$$M\left(p_h\right) = \int_{\mathcal{C}\times\mathbb{R}^3} f\left(k, p_h\right) dk^0 d^3k. \tag{78}$$

If the integrand $f\left(k, p_h\right)$ is analytic, which will be always assumed hereafter, the integral on $\mathcal{C}$ equals the integral along the imaginary axis, plus the contributions of the residues of $f\left(k, p_h\right)$ at each pole that has passed through $\mathcal{I}$ when $p_h^0 \to E_h \in \mathbb{R}_0^+$, that is,

若被积函数 $f\left(k, p_h\right)$ 解析 (下文始终默认这一条件)，则 $\mathcal{C}$ 上的积分等于沿虚轴的积分加上当 $p_h^0 \to E_h \in \mathbb{R}_0^+$ 时所有穿过 $\mathcal{I}$ 的极点处的留数贡献，即:

$$M\left(p_h\right) = \int_{\mathcal{I}\times\mathbb{R}^3} f\left(k, p_h\right) dk^0 d^3k \pm (2\pi i) \int_{\mathbb{R}^3} \sum_l \mathrm{Res}\left\{f\left(k, p_h\right), \bar{k}_l^0\left(\vec{k}, p_h\right)\right\} d^3k,$$

(79)

where the sum on the index l in the second integral in (79) is limited to those poles that crossed $\mathcal{I}$. The plus sign in (79) corresponds to poles that pass through $\mathcal{I}$ from left to right, while the minus sign corresponds to poles that move from right to left.

式 (79) 第二项积分中仅对所有穿过 $\mathcal{I}$ 的极点按指标 l 求和。式 (79) 中的正号对应从左向右穿过 $\mathcal{I}$ 的极点，负号对应从右向左移动的极点。

Indeed, in order to evaluate the complex amplitude (53), we can write $\mathcal{M}$ as

实际上，为计算复振幅 (53)，我们可将 $\mathcal{M}$ 写为

$$\mathcal{M} = -\frac{\lambda^V}{S_\#}\int_{(\mathcal{C}_L\times\mathbb{R}^3)}\frac{id^4k_L}{(2\pi)^4}\frac{F_L(k_L,p_h)}{k_L^2-m^2+i\varepsilon}, \tag{80}$$

where we have omitted the indices a and b again, and we have defined

此处我们再次省略了指标 a 和 b ，并定义:

$$F_L(k_L,p_h) = \int_{(\mathcal{C}_{L-1}\times\mathbb{R}^3)}\int_{(\mathcal{C}_{L-2}\times\mathbb{R}^3)}\cdots\int_{(\mathcal{C}_1\times\mathbb{R}^3)}\prod_{i=1}^{L-1}\frac{id^4k_i}{(2\pi)^4}\frac{1}{k_i^2-m^2+i\varepsilon}$$
$$\times\prod_{j=1}^{I-L}\frac{1}{q_j^2-m^2+i\varepsilon}B(k_i,p_h) \tag{81}$$

Proceeding iteratively, we define

通过迭代推进，我们定义

$$F_L(k_L,p_h) = \int_{(\mathcal{C}_{L-1}\times\mathbb{R}^3)}\frac{id^4k_{L-1}}{(2\pi)^4}\frac{F_{L-1}(k_L,k_{L-1},p_h)}{k_{L-1}^2-m^2+i\varepsilon}, \tag{82}$$

with

其中

$$F_{L-1}(k_L,k_{L-1},p_h) = \int_{(\mathcal{C}_{L-2}\times\mathbb{R}^3)}\frac{id^4k_{L-2}}{(2\pi)^4}\frac{F_{L-2}(k_L,k_{L-1},k_{L-2},p_h)}{k_{L-2}^2-m^2+i\varepsilon}, \tag{83}$$

and so on, until the last expressions for F_2 and F_1 , namely

依此类推，直到得到 F_2 和 F_1 的最终表达式，即

$$F_2(k_L,k_{L-1},\ldots,k_2,p_h) = \int_{(\mathcal{C}_1\times\mathbb{R}^3)}\frac{id^4k_1}{(2\pi)^4}\frac{F_1(k_L,k_{L-1},\ldots,k_1,p_h)}{k_1^2-m^2+i\varepsilon}, \tag{84}$$

$$F_1(k_L,k_{L-1},\ldots,k_1,p_h) = \prod_{j=1}^{I-L}\frac{1}{q_j^2-m^2+i\varepsilon}B(k_i,p_h). \tag{85}$$

Equations (81)-(86) can be recast by means of the recursive relation

式 (81)-(86) 可以通过递推关系改写为

$$F_{i+1}\left(k_L, k_{L-1}, \dots, k_{i+1}, p_h\right) = \int_{\left(\mathcal{C}_i \times \mathbb{R}^3\right)} \frac{id^4 k_i}{(2\pi)^4} \frac{F_i\left(k_L, k_{L-1}, \dots, k_i, p_h\right)}{k_i^2 - m^2 + i\varepsilon}.$$

(86)

Therefore, the complex amplitude $\mathcal{M}$ can be obtained by evaluating the functions $F_i\left(k_L, \dots, k_i, p_h\right)$ by means of the integration formula (79), starting from $F_2\left(k_L, k_{L-1}, \dots, k_2\right)$, and then proceeding iteratively, so that

因此，复振幅 $\mathcal{M}$ 可通过积分公式 (79) 计算函数 $F_i\left(k_L, \dots, k_i, p_h\right)$ 得到，从 $F_2\left(k_L, k_{L-1}, \dots, k_2\right)$ 出发，再通过迭代过程得到:

$$F_{i+1}\left(k_L, \dots, k_{i+1}, p_h\right) = \int_{\left(\mathcal{J} \times \mathbb{R}^3\right)} \frac{id^4 k_i}{(2\pi)^4} \frac{F_i\left(k_L, \dots, k_i, p_h\right)}{k_i^2 - m^2 + i\varepsilon}$$
$$\pm \int_{\left(\mathbb{R}^3\right)} \frac{id^3 k_i}{(2\pi)^4} (2\pi i) \sum_l \operatorname{Res}\left\{\frac{F_i\left(k_L, \dots, k_i, p_h\right)}{k_i^2 - m^2 + i\varepsilon}, \bar{k}_{i,l}^0\right\},$$

(87)

$$\mathcal{M} = -\frac{\lambda^V}{S_\#}\left[\int_{\left(\mathcal{J} \times \mathbb{R}^3\right)} \frac{id^4 k_L}{(2\pi)^4} \frac{F_L\left(k_L, p_h\right)}{k_L^2 - m^2 + i\varepsilon}\right.$$
$$\pm \int_{\left(\mathbb{R}^3\right)} \frac{id^3 k_L}{(2\pi)^4} (2\pi i) \sum_l \operatorname{Res}\left\{\frac{F_L\left(k_L, p_h\right)}{k_L^2 - m^2 + i\varepsilon}, \bar{k}_{L,l}^0\right\}, \tag{88}$$

where $\bar{k}_{i,l}^0 \equiv \bar{k}_{i,l}^0\left(\vec{k}_i, p_h\right)$ are the poles of $F_i\left(k_i, p_h\right)$ in the k_i^0 complex plane, and the sum in l is extended only to those poles passing through $\mathcal{J}$ in the limit $p_h^0 \to E_h \in \mathbb{R}_0^+$. The analogue of (88) in the case of the amplitudes (23) and (63) is given by (34) and (70), respectively.

其中 $\bar{k}_{i,l}^0 \equiv \bar{k}_{i,l}^0\left(\vec{k}_i, p_h\right)$ 是 $F_i\left(k_i, p_h\right)$ 在 k_i^0 复平面上的极点，l 中的求和仅涵盖在极限 $p_h^0 \to E_h \in \mathbb{R}_0^+$ 下穿过 $\mathcal{J}$ 的极点。振幅 (23) 和 (63) 对应的 (88) 式分别由 (34) 和 (70) 给出。

From (88), replacing the explicit expressions of the residues, and making use of (36) and (40), one obtains

从 (88) 出发，代入留数的显式表达式并利用 (36) 和 (40)，可得

$$\lim_{\varepsilon \to 0} \mathcal{M}\left(E_h, \varepsilon\right) - \mathcal{M}\left(E_h, \varepsilon\right)^*$$
$$= -\frac{\lambda^V}{S_\#} \sum \int_{\Omega_1} \cdots \int_{\Omega_L} \prod_{i=1}^{L} \frac{id^4 k_i}{(2\pi)^4} \prod_{k=1}^{N} (-2\pi i)\, \delta\left(Q_k^2 - m^2\right) \sigma\left(Q_k^0\right)$$
$$\times \prod_{j=1}^{I-N} \frac{1}{Q_j^2 - m^2 + i\varepsilon} B\left(k_i, p_h\right), \tag{89}$$

where the Q_k are the momenta corresponding to internal lines.

其中 Q_k 是对应内线的动量。

Equation (89) expresses the Cutkosky rules for the nonlocal scalar theory. Each term in the sum in (89) corresponds to a cut diagram in which N propagators are on-shell, while $I-N$ are not on-shell. Moreover, for each term, the i -th integration region Ω_i can be $\mathbb{R}^4$ or $\mathcal{I}\times\mathbb{R}^3$, depending whether the corresponding loop momentum k_i is contained in the propagator of one of the cut lines or not.

(89) 式给出了非局部标量理论的 Cutkosky 规则。(89) 求和中的每一项对应一个切割图，其中 N 个传播子在壳，而 $I-N$ 不在壳。此外，对于每一项，i 次积分区域 Ω_i 可以是 $\mathbb{R}^4$ 或 $\mathcal{I}\times\mathbb{R}^3$ ，具体取决于对应的圈动量 k_i 是否包含在某条切割线的传播子中。

Still, (89) does not specify which cut diagrams must be included in the sum. If, for instance, some cut diagrams corresponding to anomalous thresholds were included in the sum, or some of the diagrams corresponding to normal thresholds were absent, we should conclude that (14) is violated and the theory is not unitary. In principle, it is an hard task to determine which cut diagrams actually contribute to (89). In fact, in order to write (88), one has to find the poles of the functions $F_i(k_L,k_{L-1},\ldots,k_i,p_h)$ solving the Landau equations, and determine which ones pass through the imaginary axis of the complex k_i^0 plane when we take the limit of real external energies; and this analysis must be repeated variable by variable.

但 (89) 并未指明求和中需要包含哪些切割图。例如，若求和中包含了反常阈值对应的切割图，或是缺失了部分正常阈值对应的切割图，我们就可以得出结论:(14) 式不成立，该理论不是幺正的。原则上，确定哪些切割图对 (89) 有贡献是一项十分困难的任务。因为要写出 (88)，必须先求出函数 $F_i(k_L,k_{L-1},\ldots,k_i,p_h)$ 的极点 (求解朗道方程)，再确定当我们取实外能极限时哪些极点会穿过复 k_i^0 平面的虚轴；而且该分析必须对每个变量逐一重复进行。

However, we can circumvent this difficulty and prove the unitarity of the theory as follows: Let us consider the local version of the action (1) and (5), by setting $H(-\sigma\Box)\equiv 0$, that is,

不过我们可以绕过这一困难，按如下方法证明理论的幺正性: 令 $H(-\sigma\Box)\equiv 0$ ，考虑作用量 (1) 和 (5) 的局部形式，即

$$\mathcal{L}_\phi=\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}m^2\varphi^2-\lambda\sum_{n=4}^{N}\frac{c_n}{n!}\varphi^n. \tag{90}$$

Considering a generic amplitude $\mathcal{M}$, it is easy to show that all and only the cut diagram that contribute to $\mathcal{M}-\mathcal{M}^*$ in the case of the local theory (90) contribute to (89) in the case of the nonlocal theory. The proof is obtained, noting that one can repeat all the steps that we followed for finding (89) also for the local theory (90), which can be defined in Euclidean signature and then continued to real external energies. So that, for the local theory one has

考虑任意振幅 $\mathcal{M}$ ，不难证明: 在非局部理论中，对 (89) 有贡献的切割图，恰好就是局部理论 (90) 中对 $\mathcal{M}-\mathcal{M}^*$ 有贡献的那些切割图。证明思路如下: 我们可以将推导 (89) 的全部步骤在局部理论 (90) 中重复一遍，局部理论 (90) 可以先在欧几里得号差下定义，再解析延拓到实外能，因此局部理论也有

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\mathcal{M}\left(E_h,\varepsilon\right)-\mathcal{M}(E_h,\varepsilon)^* \\
&= -\frac{\lambda^V}{S_\#}\sum\int_{\Omega_1}\cdots\int_{\Omega_L}\prod_{i=1}^{L}\frac{id^4k_i}{(2\pi)^4}\prod_{k=1}^{N}(-2\pi i)\,\delta\left(Q_k^2-m^2\right)\sigma\left(Q_k^0\right)\times \\
&\times\prod_{j=1}^{I-N}\frac{1}{Q_j^2-m^2+i\varepsilon}.
\end{aligned}\tag{91}
$$

Since the local and nonlocal amplitudes have the same singularities, then (91) contains the same terms with the same delta functions as in (89), with the only difference that $B\left(k_i,p_h\right)\equiv 1$ in the local case. Since the local theory (90) is unitary, the sum (91) does not contain contributions from cut diagrams corresponding to anomalous thresholds. Indeed, the same will be true for (89) in the case of our nonlocal theory. Furthermore, the unitarity of (90) also implies that (91) contains contributions from all the cut diagrams that appear in the sum in (14), so that the same will be for the nonlocal theory.

由于局部振幅和非局部振幅具有相同的奇点，因此 (91) 和 (89) 包含相同的项与相同的 δ 函数，唯一区别是局部情形下为 $B\left(k_i,p_h\right)\equiv 1$ 。因为局部理论 (90) 是幺正的，所以求和 (91) 不包含反常阈值切割图的贡献，由此可知我们的非局部理论的 (89) 也同样不包含反常阈值贡献。此外，(90) 的幺正性还意味着 (91) 包含了 (14) 求和中所有切割图的贡献，因此非局部理论也满足这一点。

Therefore, we conclude that the cut diagrams that contribute to (89) in the nonlocal case are all and only those that contribute to (91) in the local case. This fact, together with the validity of the Cutkosky rules and the unitarity of the local theory (90), proves the unitarity of the nonlocal field theory.

因此我们得出结论: 非局域情形下对 (89) 有贡献的切割图，恰好就是局域情形下对 (91) 有贡献的全部切割图。这一事实，加上卡特科斯基规则的有效性以及局域理论的幺正性 (90)，证明了非局域场论的幺正性。

Non-unitarity of the Minkowskian Theory

闵氏理论的非幺正性

In the previous sections, we have shown that, if one defines the nonlocal scalar theory in Euclidean signature, assuming that all the internal and external energies k^0 and p^0 are purely imaginary, and then continuing the scattering amplitudes to real positive external energies, (14) is fulfilled and the theory is unitary. Here we want to show that this procedure is necessary to ensure the unitarity. In fact, if one defines the quantum theory directly in Minkowskian space, taking $k^0\in\mathbb{R}$ and $p^0\in\mathbb{R}_0^+$ from the beginning, the imaginary part of the scattering amplitudes is no longer given by the Cutkosky rules, but it contains some extra terms that spoil the unitarity of the theory, see [30].

在前文中我们已经证明: 若在欧氏号差下定义非局部标量理论，假设所有内、外能量 k^0 和 p^0 均为纯虚数，再将散射振幅解析延拓到正实外能量，则条件 (14) 成立，理论具有幺正性。本文将证明上述步骤是保证幺正性的必要条件: 如果直接在闵氏空间定义量子理论，从一开始就取 $k^0 \in \mathbb{R}$ 和 $p^0 \in \mathbb{R}_0^+$，则散射振幅的虚部不再由卡特科斯规则给出，而是会出现破坏理论幺正性的额外项，参见文献 [30]。

In order to prove this statement, we consider the same one-loop diagram in Fig. 1 that we have analyzed in section "One-Loop Diagram" in Euclidean space, corresponding to the one-loop contribution to the scattering $\phi + \phi \to \phi + \phi$ in the nonlocal scalar field with theory (1) or (5) with quartic interaction. To enforce the convergence of the loop integrals in both Euclidean and in Minkowskian signatures, we impose that $H\left(\sigma p^2\right)$ depends on the momentum via $\left(\sigma k^2\right)^2$. Therefore, we redefine the nonlocal form factor in (1) and (5) by means of the replacement $H\left[-\sigma\Box\right] \to H\left[-\sigma^2\Box^2\right]$. Moreover, we replace (2) and (17) with the following conditions on the redefined $H(z)$:

为证明这一结论，我们沿用我们在欧氏空间“单圈图”一节中分析过的图 1 的同一个单圈图，该图对应带四次相互作用的理论 (1) 或 (5) 中非局部标量场散射 $\phi + \phi \to \phi + \phi$ 的单圈贡献。为保证欧氏和闵氏号差下圈积分的收敛性，我们要求 $H\left(\sigma p^2\right)$ 通过 $\left(\sigma k^2\right)^2$ 依赖于动量。因此我们对 (1) 和 (5) 中的非局部形状因子通过替换 $H\left[-\sigma\Box\right] \to H\left[-\sigma^2\Box^2\right]$ 重新定义。此外，我们将 (2) 和 (17) 替换为对重新定义后的 $H(z)$ 的如下条件:

$$H\left(\sigma^2 m^4\right) = 1, \quad \text{and} \quad \lim_{z\to+\infty} e^{-H(z)} = 0. \tag{92}$$

We still assume that $\exp\left(H[z]\right)$ is an entire function without zeros at finite z , but we will see that, in the Minkowskian case, this condition is not enough to imply the unitarity of the theory.

我们仍假设 $\exp\left(H[z]\right)$ 是整函数，在有限 z 处没有零点，但我们会看到，在闵氏情形下，这一条件不足以保证理论的幺正性。

In Minkowskian signature, the one-loop amplitude under analysis reads

在闵氏号差下，本文分析的单圈振幅可以写为

$$\mathcal{M}\left(p_h, \varepsilon\right) = -\frac{i\lambda^2}{2}\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{e^{-H\left(\sigma^2 k^4\right)}}{k^2 - m^2 + i\varepsilon} \frac{e^{-H\left(\sigma^2 (k-p)^4\right)}}{(k-p)^2 - m^2 + i\varepsilon}, \tag{93}$$

where $p = p_1 + p_2 = p_3 + p_4$, and p_1, p_2, p_3, p_4 are the external momenta. The same amplitude in Euclidean signature, corresponding to the analytic continuation of (26) with the redefinition of $H(z)$, is

其中 $p = p_1 + p_2 = p_3 + p_4$ 和 p_1, p_2, p_3, p_4 是外动量。经 $H(z)$ 重新定义后，对应 (26) 解析延拓得到的欧氏号差下的同一振幅为

$$\mathcal{M}\left(p_h, \varepsilon\right) = -\frac{\lambda^2}{2}\int_{(\mathcal{C}\times\mathbb{R}^3)} \frac{id^4k}{(2\pi)^4} \frac{e^{-H\left(\sigma^2 k^4\right)}}{k^2 - m^2 + i\varepsilon} \frac{e^{-H\left(\sigma^2 (k-p)^4\right)}}{(k-p)^2 - m^2 + i\varepsilon}.$$

(94)

Again, since the form factor $\exp(H[z])$ has no zeros in the finite complex z plane, the poles of the integrand in (93) are only those of the two propagators. Therefore, the singularities of (23) are determined by the Landau equations. They occur in the limit $\varepsilon \to 0$, when the two internal lines are on-shell simultaneously, pinching the integration contour. Thus, the nonlocal Minkowskian amplitude (23) has the same singularity structure as the corresponding local amplitude. The same is of course true for (94).

同样地，由于形状因子 $\exp(H[z])$ 在有限复 z 平面内没有零点，因此 (93) 中被积函数的极点仅来自两个传播子。因此 (23) 的奇点由朗道方程决定，奇点出现在 $\varepsilon \to 0$ 极限下: 此时两条内线同时在壳，挤压了积分围道。因此，非局部闵氏振幅 (23) 的奇点结构与对应局部振幅相同，这一结论当然对 (94) 也成立。

Thanks to Lorentz invariance, we set $\vec{p} = 0$ in (93) without loss of generality. The poles of the propagators are given by (28) and (30), which we recast as

利用洛伦兹不变性，我们可以不失一般性地在 (93) 中令 $\vec{p} = 0$ 。传播子的极点由 (28) 和 (30) 给出，我们将其改写为

$$\bar{k}^0_{1,2} = \pm\sqrt{\vec{k}^2 + m^2 - i\varepsilon} \equiv \pm\omega_k, \ \bar{k}^0_{3,4} = p^0 \pm \omega_k. \tag{95}$$

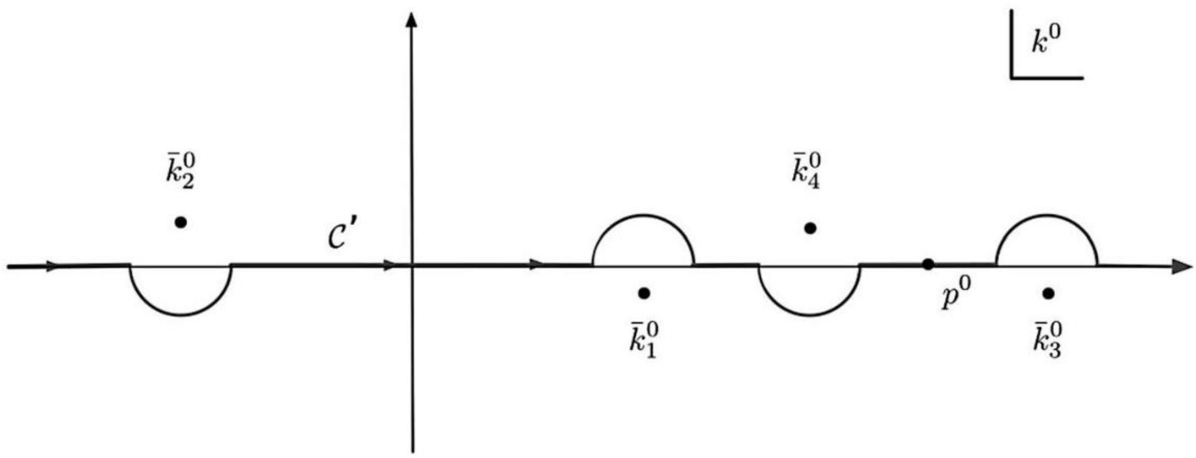


Fig. 10 We plot the poles $\bar{k}^0_1, \bar{k}^0_2, \bar{k}^0_3, \bar{k}^0_4$ in the complex k^0 plane. When $\varepsilon \to 0$, such poles become real, and the integration contour C' is obtained by deforming the real axis around them

图 10 我们绘制了复 k^0 平面内极点 $\bar{k}^0_1, \bar{k}^0_2, \bar{k}^0_3, \bar{k}^0_4$ 的位置。当 $\varepsilon \to 0$ 时，这些极点变为实极点，积分围道 C' 是通过将实轴绕极点变形得到的

These poles are plotted in Fig. 10 in the complex k^0 plane. As usual, as long as $\varepsilon > 0$, these poles are well separated, the integration contour, which coincides with the real axis, is not pinched, and the amplitude is not singular. When $\varepsilon \to 0$, the poles become real, and the integral in the k^0 variable in (93) is performed on the contour $\mathcal{C}'$ obtained by deforming the real axis around $\bar{k}^0_1, \bar{k}^0_2, \bar{k}^0_3, \bar{k}^0_4$, as depicted in Fig. 10. This corresponds to the usual Feynman prescription for Minkowskian loop integrals. It is easy to see that, in the limit $\varepsilon \to 0$, the poles $\bar{k}^0_1$ and $\bar{k}^0_4$ merge for $|\vec{k}| = \sqrt{(p^0/2)^2 - m^2}$, provided that $p^0 \geq 2m$, pinching $\mathcal{C}'$. In analogy with the case of a local scalar theory, this implies that the Minkowskian amplitude (93) has a branch-cut singularity at $p^0 \geq 2m$.

这些极点已绘制在复 k^0 平面的图 10 中。和常规情况一致，只要 $\varepsilon > 0$ ，这些极点就会充分分离，与实轴重合的积分围道不会被挤压，振幅也没有奇异性。当 $\varepsilon \to 0$ 时，极点变为实极点，式 (93) 中对变量 k^0 的积分需要沿围道 $\mathcal{C}'$ 计算，该围道是将实轴围绕 $\bar{k}_1^0, \bar{k}_2^0, \bar{k}_3^0, \bar{k}_4^0$ 形变得到的，如图 10 所示。这对应于闵氏圈积分的常规费曼 prescriptions。不难看出，在 $\varepsilon \to 0$ 极限下，若 $p^0 \geq 2m$ ，极点 $\bar{k}_1^0$ 和 $\bar{k}_4^0$ 会在 $\left|\vec{k}\right| = \sqrt{\left(p^0/2\right)^2 - m^2}$ 处重合，挤压围道 $\mathcal{C}'$ 。与局域标量理论的情况类似，这说明闵氏振幅 (93) 在 $p^0 \geq 2m$ 处存在分支切割奇点。

The integral along $\mathcal{C}'$ can be split into two contributions: one given by the principal part of the integral calculated along the real axis, and the other given by the contribution of the four infinitesimal arcs plotted in Fig. 10. As each arc gives a term $\pm\pi i$ times the residue at the corresponding pole, in the limit $\varepsilon \to 0$ the Minkowskian amplitude (93) becomes

沿 $\mathcal{C}'$ 的积分可以拆分为两部分: 一部分是沿实轴计算的积分主值，另一部分是图 10 中画出的四段无穷小圆弧的贡献。由于每段圆弧会给出一个 $\pm\pi i$ 乘以对应极点留数的项，在 $\varepsilon \to 0$ 极限下，闵氏振幅 (93) 变为

$$\mathcal{M}(p_h) \equiv \lim_{\varepsilon\to 0} \mathcal{M}(p_h, \varepsilon) = \frac{\lambda^2}{2}\left[-i\mathcal{P}\left\{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{e^{-H(\sigma^2 k^4)}}{k^2 - m^2} \frac{e^{-H\left(\sigma^2 (k-p)^4\right)}}{(k-p)^2 - m^2}\right\}\right.$$
$$\left. +\pi \lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^4} \left(\mathrm{Res}\left[\bar{k}_2^0\right] + \mathrm{Res}\left[\bar{k}_4^0\right] - \mathrm{Res}\left[\bar{k}_1^0\right] - \mathrm{Res}\left[\bar{k}_3^0\right]\right)\right., \tag{96}$$

where $\mathcal{P}$ is the principal part of the integral in the k^0 variable, and

其中 $\mathcal{P}$ 是 k^0 变量积分的主值，且

$$\mathrm{Res}\left[\bar{k}_1^0\right] = \frac{1}{2\omega_k} \left.\frac{e^{-H\left(\sigma^2 (k-p)^4\right)}}{\left((k-p)^2 - m^2 + i\varepsilon\right)^2}\right|_{k^0 = \bar{k}_1^0}, \quad \mathrm{Res}\left[\bar{k}_2^0\right] = -\frac{1}{2\omega_k} \left.\frac{e^{-H\left(\sigma^2 (k-p)^4\right)}}{\left((k-p)^2 - m^2 + i\varepsilon\right)^2}\right|_{k^0 = \bar{k}_2^0},$$
$$\mathrm{Res}\left[\bar{k}_3^0\right] = \frac{1}{2\omega_k} \left.\frac{e^{-H(\sigma^2 k^4)}}{(k^2 - m^2 + i\varepsilon)^2}\right|_{k^0 = \bar{k}_3^0}, \quad \mathrm{Res}\left[\bar{k}_4^0\right] = -\frac{1}{2\omega_k} \left.\frac{e^{-H(\sigma^2 k^4)}}{(k^2 - m^2 + i\varepsilon)^2}\right|_{k^0 = \bar{k}_4^0}. \tag{97}$$

Note that for $k^0 = \bar{k}_2^0 = -\omega_k$ one has $(k-p)^2 - m^2 \neq 0$ for any $\vec{k} \in \mathbb{R}^3$. In fact, the pole $\bar{k}_2^0$ is always far from $\bar{k}_3^0$ and $\bar{k}_4^0$. That implies that $\mathrm{Res}\left[\bar{k}_2^0\right]$ has no poles in the integration volume for $\varepsilon \to 0$. In the same way, for $k^0 = \bar{k}_3^0 = p^0 + \omega$, one has $k^2 - m^2 \neq 0$, and $\mathrm{Res}\left[\bar{k}_3^0\right]$ has no poles for $\vec{k} \in \mathbb{R}^3$ for $\varepsilon \to 0$. On the contrary, since $\bar{k}_1^0$ and $\bar{k}_4^0$ coincide in the limit $i\varepsilon \to 0$ for $\left|\vec{k}\right| = \sqrt{(p^0/2) - m^2} \equiv k_p$ when $p^0 \geq 2m$, $\mathrm{Res}\left[\bar{k}_1^0\right]$ and $\mathrm{Res}\left[\bar{k}_4^0\right]$ are singular on the 3-sphere $\left|\vec{k}\right| = k_p$. Therefore, we expect a branch-cut singularity at $p^0 = E \geq 2m$.

注意，对任意 $\vec{k}\in\mathbb{R}^3$，$k^0=\bar{k}_2^0=-\omega_k$ 满足 $(k-p)^2-m^2\neq 0$。事实上，极点 $\bar{k}_2^0$ 始终远离 $\bar{k}_3^0$ 和 $\bar{k}_4^0$。这说明 $\mathrm{Res}\left[\bar{k}_2^0\right]$ 在 $\varepsilon\to 0$ 的积分区域内没有极点。同理，对 $k^0=\bar{k}_3^0=p^0+\omega$ 有 $k^2-m^2\neq 0$，且 $\mathrm{Res}\left[\bar{k}_3^0\right]$ 在 $\varepsilon\to 0$ 的 $\vec{k}\in\mathbb{R}^3$ 处没有极点。相反，当 $p^0\geq 2m$, $\mathrm{Res}\left[\bar{k}_1^0\right]$ 和 $\mathrm{Res}\left[\bar{k}_4^0\right]$ 在三球面 $\left|\vec{k}\right|=k_p$ 上奇异时，在 $\left|\vec{k}\right|=\sqrt{(p^0/2)-m^2}\equiv k_p$ 的极限 $i\varepsilon\to 0$ 下 $\bar{k}_1^0$ 与 $\bar{k}_4^0$ 重合。因此我们预期在 $p^0=E\geq 2m$ 处存在支割奇点。

We can rearrange (96) using (36), so that

我们可以利用式 (36) 改写式 (96)，得到

$$\mathcal{M}(p_h)=\frac{\lambda^2}{2}\left[-i\mathcal{P}\left\{\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2}\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2}\right\}+\right.$$
$$-\pi\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3}\frac{d^4k}{(2\pi)^4}\left(\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2+i\varepsilon}\delta\left((k-p)^2-m^2\right)\right.$$
$$\left.\left.+\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2+i\varepsilon}\delta\left(k^2-m^2\right)\right)\right], \tag{98}$$

and using the relation (40), we obtain

再利用关系式 (40)，我们得到

$$\mathcal{M}(p_h)=\frac{\lambda^2}{2}\left[-i\mathcal{P}\left\{\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2}\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2}\right\}\right.$$
$$-\frac{i}{2}(2\pi i)^2\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\delta\left((k-p)^2-m^2\right)\delta\left(k^2-m^2\right)+$$
$$-\pi\mathcal{P}\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\left(\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2}\delta\left((k-p)^2-m^2\right)\right.$$
$$\left.\left.+\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2}\delta\left(k^2-m^2\right)\right)\right]. \tag{99}$$

We note that the last term in (99) is real, while the first two terms are purely imaginary, so that

我们注意到式 (99) 的最后一项是实项，而前两项是纯虚项，因此

$$\mathcal{M}(p_h)-\mathcal{M}^*(p_h)=\lambda^2\left[-i\mathcal{P}\left\{\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2}\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2}\right\}\right.$$
$$\left.-\frac{i}{2}(2\pi i)^2\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\delta\left((k-p)^2-m^2\right)\delta\left(k^2-m^2\right)\right].$$

(100)

A simple comparison between (48) and (100) shows that the Cutkosky rules are valid if and only if the first integral in (100) is identically zero for any σ, p and m. As we will show below, this is not the case, indeed the Cutkosky rules are no longer valid, and the nonlocal Minkowskian theory is not unitary.

对式 (48) 和式 (100) 简单对比后可知，当且仅当式 (100) 中第一个积分对任意 σ, p 和 m 都恒为零时，Cutkosky 规则才成立。正如我们下文将要证明的，事实并非如此，Cutkosky 规则实际上不再成立，非定域闵氏理论不是幺正的。

Let us recast such integral performing the translation $k \to k + p/2$, so that it will be equal to the function $I\left(\sigma, \left(p^0\right)^2, m^2\right)$ defined as

我们对该积分做变换 $k \to k + p/2$ 后重写，它将等于如下定义的函数 $I\left(\sigma, \left(p^0\right)^2, m^2\right)$

$$I\left(\sigma, \left(p^0\right)^2, m^2\right) \equiv \mathcal{P}\left\{\int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{e^{-H\left(\sigma^2\left(k+\frac{p}{2}\right)^4\right)}}{\left(k+\frac{p}{2}\right)^2 - m^2} \frac{e^{-H\left(\sigma^2\left(k-\frac{p}{2}\right)^4\right)}}{\left(k-\frac{p}{2}\right)^2 - m^2}\right\}. \tag{101}$$

Indeed, the Cutkosky rules would be valid if and only if $I\left(\sigma, \left(p^0\right)^2, m^2\right) \equiv 0$ for any σ, p^0 and m^2 . Note that the integrand in (101) is even in p , so that I must be a function of the scalar $p^2 = \left(p^0\right)^2$, as it must in virtue of the Lorentz invariance. Furthermore, $I\left(\sigma, \left(p^0\right)^2, m^2\right)$ is also even in σ . The integrand in (101) is also even in k^0 , but this fact does not determine any specific property of $I\left(\sigma, \left(p^0\right)^2, m^2\right)$.

事实上，当且仅当对任意 σ, p^0 和 m^2 都满足 $I\left(\sigma, \left(p^0\right)^2, m^2\right) \equiv 0$ 时，Cutkosky 规则才成立。注意式 (101) 中的被积函数关于 p 是偶函数，因此根据洛伦兹不变性，I 必须是标量 $p^2 = \left(p^0\right)^2$ 的函数。此外，$I\left(\sigma, \left(p^0\right)^2, m^2\right)$ 关于 σ 也是偶函数。式 (101) 中的被积函数关于 k^0 同样是偶函数，但这一性质不会限定 $I\left(\sigma, \left(p^0\right)^2, m^2\right)$ 的任何特殊性质。

Let us derive some general features of (101). First, note that for a local theory, corresponding to $\sigma = 0$, this integral is zero. In fact, in this case, the exponential functions in (101) are constant, and the integrand reduces to

我们来推导式 (101) 的一些一般性质。首先注意，对于对应 $\sigma = 0$ 的定域理论，该积分为零。实际上，在这种情况下，式 (101) 中的指数函数是常数，被积函数化简为

$$\begin{aligned}
&\frac{e^{-H(0)}}{\left(k+\frac{p}{2}\right)^2 - m^2} \frac{e^{-H(0)}}{\left(k-\frac{p}{2}\right)^2 - m^2} \\
&= \frac{e^{-2H(0)}}{4\omega^2}\left[\left(\frac{1}{p^0} - \frac{1}{p^0 + 2\omega}\right)\left(\frac{1}{k^0 - \frac{p^0}{2} - \omega} - \frac{1}{k^0 + \frac{p^0}{2} + \omega}\right)\right. \\
&\quad \left. + \left(\frac{1}{p^0} - \frac{1}{p^0 - 2\omega}\right)\left(\frac{1}{k^0 - \frac{p^0}{2} + \omega} - \frac{1}{k^0 + \frac{p^0}{2} - \omega}\right)\right],
\end{aligned} \tag{102}$$

where $\omega\left(\vec{k}\right) = \sqrt{\vec{k}^2 + m^2}$, and indeed, the integration in the k^0 variable in (101) reduces to a linear combination of integrals of the type

其中 $\omega\left(\vec{k}\right) = \sqrt{\vec{k}^2 + m^2}$ ，并且不难看出，式 (101) 中对 k^0 变量的积分可化简为以下类型积分的线性组合

$$\mathcal{P}\left\{\int_{\mathbb{R}} dk^0 \frac{1}{k^0 \pm \left(\frac{p^0}{2} \pm \omega\left(\vec{k}\right)\right)}\right\} = 0 \tag{103}$$

that are all null, because the principal part is taken integrating in an interval symmetric with respect to the pole. Indeed, in the local case $I\left(\sigma = 0, p^0, m^2\right) = 0$ by symmetry, the Cutkosky rules are valid and the Minkowskian theory is unitary. This is what we expected, as the local Euclidean and Minkowskian theories are equivalent. From these considerations, we infer that it must be

它们全部为零，因为主值积分是在关于极点对称的区间上进行的。事实上，定域情形下 $I\left(\sigma = 0, p^0, m^2\right) = 0$ 由对称性满足，卡茨科夫斯基规则成立，闵氏理论是幺正的。这符合我们的预期，因为定域欧氏理论与定域闵氏理论是等价的。基于上述考虑，我们可以推得必定有

$$\lim_{\sigma\to 0} I\left(\sigma, \left(p^0\right)^2, m^2\right) = 0,\ \sigma \in \mathbb{R}, \forall \left(p^0\right)^2, m^2 \in \mathbb{R}. \tag{104}$$

Moreover, by means of the change of variables $k \to k/\sqrt{|\sigma|}$ in the defining expression (101), it is easy to show that

此外，通过对定义式 (101) 做变量替换 $k \to k/\sqrt{|\sigma|}$ ，容易证明

$$I\left(\sigma, \left(p^0\right)^2, m^2\right) = I\left(\sigma = 1, |\sigma|\left(p^0\right)^2, |\sigma|\, m^2\right) \equiv \phi\left(|\sigma|\left(p^0\right)^2, |\sigma|\, m^2\right)), \tag{105}$$

so that I depends on σ through the variables $|\sigma|\left(p^0\right)^2$ and $|\sigma|\, m^2$, which is a consequence of the fact that I is dimensionless. From (105) we also learn that I is not an analytic function of σ in $\sigma = 0$, since it depends on $|\sigma|$. Form (101) it is also easy to see that

因此 I 通过变量 $|\sigma|\left(p^0\right)^2$ 和 $|\sigma|\, m^2$ 依赖于 σ ，这是 I 无量纲这一事实的推论。从式 (105) 我们还可知，I 在 $\sigma = 0$ 中不是 σ 的解析函数，因为它依赖于 $|\sigma|$ 。从式 (101) 还容易看出

$$I\left(\sigma, \left(ap^0\right)^2, m^2\right) = I\left(a^2\sigma, \left(p^0\right)^2, m^2/a^2\right), \tag{106}$$

so that, using (104), one has

因此，结合式 (104) 可得

$$\lim_{a\to 0} I\left(\sigma, \left(ap^0\right)^2, m^2\right) = \lim_{a\to 0} I\left(a^2\sigma, \left(p^0\right)^2, m^2/a^2\right) = 0\ \forall m, \tag{107}$$

which means that

这意味着

$$I\left(\sigma,\left(p^0\right)^2=0,m^2\right)=\phi\left(|\sigma|\left(p^0\right)^2=0,|\sigma|\,m^2\right)=0\ \forall\ m. \tag{108}$$

These relations imply that I must have the form

这些关系表明 I 必定具有如下形式

$$I\left(\sigma,\left(p^0\right)^2,m^2\right)=f\left(|\sigma|\left(p^0\right)^2\right)\times\psi\left(|\sigma|\left(p^0\right)^2,|\sigma|\,m^2\right),$$

(109)

with $f(0)=0,\psi(0,0)\neq 0$.

Regardless of the details of the functions $f\left(|\sigma|\left(p^0\right)^2\right)$ and $\psi\left(|\sigma|\left(p^0\right)^2\right),|\sigma|\,m^2$), one can prove that $I\left(\sigma,\left(p^0\right)^2,m^2\right)$ ⍰ 0 . Let us express I as a function of $\alpha=\sigma^2\in\ \mathbb{R}_0^+$, so that

无论函数 $f\left(|\sigma|\left(p^0\right)^2\right)$ 和 $\psi\left(|\sigma|\left(p^0\right)^2\right),|\sigma|\,m^2$ 的具体形式如何，都可以证明 $I\left(\sigma,\left(p^0\right)^2,m^2\right)$ ⍰ 0。我们将 I 表示为 $\alpha=\sigma^2\in\ \mathbb{R}_0^+$ 的函数，于是有

$$\tilde{I}\left(\alpha,\left(p^0\right)^2,m^2\right)\equiv I\left(\sqrt{\alpha},\left(p^0\right)^2,m^2\right)=\phi\left(\sqrt{\alpha}\left(p^0\right)^2,\sqrt{\alpha}m^2\right)$$

$$=\mathcal{P}\left\{\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-H\left(\alpha(k+p/2)^4\right)}}{(k+p/2)^2-m^2}\frac{e^{-H\left(\alpha(k-p/2)^4\right)}}{(k-p/2)^2-m^2}\right\}. \tag{110}$$

Since $H(z)$ is entire, it must be $H(z)=H(0)+\sum\limits_{n=1}^{\infty}c_nz^n$. Let us assume $H'(0)=\ c_1\neq 0$. In order to show that $\tilde{I}\left(\alpha,\left(p^0\right)^2,m^2\right)$ ⍰ 0 , it is sufficient to show that its first derivative is not identically zero. One has

由于 $H(z)$ 是整函数，因此必定有 $H(z)=H(0)+\sum\limits_{n=1}^{\infty}c_nz^n$ 。我们假设 $H'(0)=\ c_1\neq 0$。要证明 $\tilde{I}\left(\alpha,\left(p^0\right)^2,m^2\right)$ ⍰ 0，只需证明它的一阶导数不恒为零，可得

$$\partial_\alpha\tilde{I}\left(\alpha,\left(p^0\right)^2,m^2\right)=-\mathcal{P}\left\{\int_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^4}\frac{e^{-H\left(\alpha(k+p/2)^4\right)}}{(k+p/2)^2-m^2}\frac{e^{-H\left(\alpha(k-p/2)^4\right)}}{(k-p/2)^2-m^2}\right.$$

$$\left.\times\left[H'\left(\alpha(k+p/2)^4\right)(k+p/2)^4+H'\left(\alpha(k-p/2)^4\right)(k-p/2)^4\right]\right\},$$

(111)

so that

因此

$$\lim_{\alpha\to 0^+}\left|\partial_\alpha\tilde{I}\left(\alpha,\left(p^0\right)^2,m^2\right)\right|$$

$$= \left| H'(0)\, e^{-2H(0)} \mathcal{P} \left\{ \int_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^4} \frac{(k+p/2)^4 + (k-p/2)^4}{\left((k-p/2)^2 - m^2\right)\left((k+p/2)^2 - m^2\right)} \right\} \right| = +\infty,$$

(112)

as the last integral is infinite, since its integrand goes to 1 when $k^2 \to \pm\infty$. Indeed, if $H'(0) \neq 0$,(112) implies that the first term in (100) is not identically zero. If, instead, the function $H(z)$ has the first $m-1$ derivatives null, so that $H(z) = H(0) + z^m \left(\sum_{n=0}^{\infty} c_{n+m} z^n \right)$ with $c_m \neq 0$, one can define $\alpha = \sigma^{2m}$ and proceed in the same way to show that $\left| \partial_\alpha I\left((\alpha)^{1/2m}, \left(p^0\right)^2, m^2 \right) \right| \to +\infty$ for $\alpha \to 0^+$. Indeed, unless $H \equiv 0$ that corresponds to the local theory, $I\left(\sigma, \left(p^0\right)^2, m^2\right)$ cannot be identically zero.

由于最后一个积分是发散的，因为当 $k^2 \to \pm\infty$ 时，被积函数趋近于 1。事实上，若 $H'(0) \neq 0$ ，式 (112) 表明式 (100) 中的第一项并不恒为零。反之，如果函数 $H(z)$ 的前 $m-1$ 阶导数都为零，即满足 $H(z) = H(0) + z^m \left(\sum_{n=0}^{\infty} c_{n+m} z^n \right)$ 且 $c_m \neq 0$ ，我们可以定义 $\alpha = \sigma^{2m}$ 并按同样的方法推导，证明对于 $\alpha \to 0^+$ 有 $\left| \partial_\alpha I\left((\alpha)^{1/2m}, \left(p^0\right)^2, m^2 \right) \right| \to +\infty$ 成立。事实上，除非对应局域理论的 $H \equiv 0$ 成立，否则 $I\left(\sigma, \left(p^0\right)^2, m^2\right)$ 不可能恒为零。

Therefore, for a nonlocal theory with $H(z)$ ⍰ 0 defined in Minkowskian space, the first term in (100) is not identically zero, and indeed the imaginary part of the complex amplitude (93) is not given by the Cutkosky rules, so that (14) is violated and the theory is not unitary.

因此，对于定义在闵氏空间中带有 $H(z)$ ⍰ 0 的非局域理论，式 (100) 中的第一项并不恒为零，且复振幅 (93) 的虚部确实不满足卡特斯基规则，因此式 (14) 不成立，该理论不是幺正的。

We conclude this section by discussing the relation between the Euclidean and Minkowskian scalar theories. First of all, we stress that not all the nonlocal theories defined in Euclidean signature can be defined also in Minkowskian space. For instance, if we choose $H(z) = -z$, the Euclidean amplitude (26) converges, while in Minkowskian signature the term $e^{\sigma k^2}$ diverges exponentially for $k^0 \to \pm\infty$ and converges for $\left|\vec{k}\right| \to +\infty$, so that the corresponding Minkowskian amplitude is not defined. If, on the contrary, we take the nonlocal form factor as $e^{H[-\sigma^2 \Box^2]}$ with the condition (92), for instance, choosing $H\left(\sigma^2 k^4\right) = \sigma^2 k^4$, both the Euclidean (93) and the Minkowskian (94) amplitudes are well defined.

我们在本节最后讨论欧氏标量理论与闵氏标量理论之间的关系。首先我们要强调，并非所有欧氏号差下定义的非局域理论都能在闵氏空间中定义。例如，若我们选取 $H(z) = -z$ ，欧氏振幅 (26) 收敛，但在闵氏号差下，项 $e^{\sigma k^2}$ 当 $k^0 \to \pm\infty$ 时指数发散，仅当 $\left|\vec{k}\right| \to +\infty$ 时收敛，因此对应的闵氏振幅不存在。反之，如果我们将非局域形状因子取为满足条件 (92) 的 $e^{H[-\sigma^2 \Box^2]}$ ，例如选取 $H\left(\sigma^2 k^4\right) = \sigma^2 k^4$ ，则欧氏振幅 (93) 和闵氏振幅 (94) 都是良好定义的。

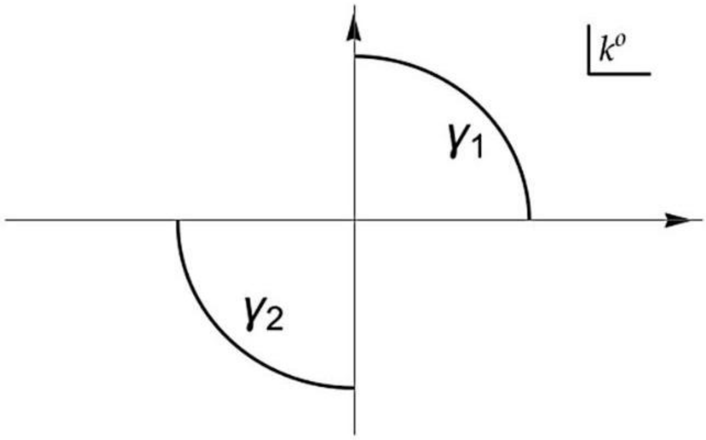
From (93) and (94) we see that the fundamental difference between the Minkowskian and Euclidean amplitudes stays in the integration contour. In the Euclidean case, the contour $\mathcal{C}$ is plotted in Fig. 3, while the contour $\mathcal{C}'$ for the Minkowskian amplitude is plotted in Fig. 10. Using the residue theorem, one can prove that the difference between (93) and (94) is given by the same integral calculated along the two arches γ_1 and γ_2 of the complex k^0 plane plotted in Fig. 11, in the limit in which their radius r is sent to infinity, that is,

从式 (93) 和式 (94) 中我们可以看出，闵氏振幅和欧氏振幅的根本区别在于积分围道。欧氏情形下，围道 $\mathcal{C}$ 如图 3 所示，而闵氏振幅的围道 $\mathcal{C}'$ 如图 10 所示。利用留数定理可以证明，在半径 r 趋于无穷大的极限下，式 (93) 与式 (94) 的差等于沿复 k^0 平面上两个弧段 γ_1 和 γ_2 (图 11 已标出) 计算的同一个积分，即

$$\lim_{r\to+\infty}\left[\frac{i\lambda^2}{2}\int_{\gamma_1\times\mathbb{R}^3}\frac{d^4k}{(2\pi)^4}\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2+i\varepsilon}\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2+i\varepsilon}\right.$$
$$\left.-\frac{i\lambda^2}{2}\int_{\gamma_2\times\mathbb{R}^3}\frac{d^4k}{(2\pi)^4}\frac{e^{-H(\sigma^2k^4)}}{k^2-m^2+i\varepsilon}\frac{e^{-H\left(\sigma^2(k-p)^4\right)}}{(k-p)^2-m^2+i\varepsilon}\right]. \tag{113}$$

Fig. 11 We plot the arches γ_1 and γ_2 in the complex k^0 plane

图 11 我们画出了复 k^0 平面上的弧段 γ_1 和 γ_2



It is well known that, in the local case, the Euclidean and Minkowskian amplitudes (93) and (94) are the same, as they are related by a Wick rotation. In fact, when $H(z) = 0$, the integrand in (113) goes to zero faster than r when $r \to +\infty$, so that the limit (113) is zero. However, this is not the case of the nonlocal theory. In fact, the nonlocal form factor is an entire function, so that it has no poles in the finite complex plane. Nevertheless, according to the Liouville theorem, it must go to infinity for $|z| \to +\infty$ in some conical region of the complex z plane, unless it is a constant, that is, unless the theory is local. For instance, for $H\left(\sigma^2k^4\right) = \sigma^2k^4$, the nonlocal form factor diverges for $r \to +\infty$ on the portions of the arches with $\pi/8 + m\pi < \arg\left\{k^0\right\} < 3\pi/8 + m\pi$, for $m = 0, 1$. Therefore, the integrals along the arches are not zero, (93) and (94) are different, and the Euclidean and Minkowskian formulations of the nonlocal theory are inequivalent.

众所周知，在局域情况下，通过威克转动联系的欧几里得振幅与闵氏振幅 (即式 (93) 与式 (94)) 是相同的。事实上，当 $H(z) = 0$ 时，若 $r \to +\infty$ ，式 (113) 中的被积函数衰减速度快于 r ，因此极限 (113) 为零。但非局域理论并非如此。非局域形状因子是整函数，因此在有限复平面没有极点。然而根据刘维尔定理，除非它是常数 (即理论是局域的)，否则在复 z 平面的某个锥形区域中，当 $|z| \to +\infty$ 时，该函数必然趋于无穷。例如，对于 $H\left(\sigma^2k^4\right) = \sigma^2k^4$ ，当 $m = 0, 1$ ， $r \to +\infty$ 落在拱线上满足 $\pi/8 + m\pi < \arg\left\{k^0\right\} < 3\pi/8 + m\pi$ 的部分时，非局域形状因子发散。因此，沿拱线的积分不为零，式 (93) 与式 (94) 不相等，非局域理论的欧几里得表述和闵可夫斯基表述并不等价。

Therefore, in the light of these considerations, it is not surprising that only the Euclidean nonlocal theory is unitary, as the Euclidean and Minkowskian formulations have nothing to do with each other. Indeed, only the choice of the integration contours $\mathcal{C}_i$ corresponding to the Euclidean signature allows to formulate for a unitary theory, while the contours that arise in the Minkowskian signature lead to an inconsistency of the theory.

因此，根据上述结论，既然欧几里得表述和闵氏表述完全无关，只有欧几里得非局域理论满足幺正性也就不足为奇了。事实上，只有对应欧几里得号差的积分围道 $\mathcal{C}_i$ 可以构造出幺正理论，而闵氏号差下得到的围道会导致理论不自洽。

Cross-References

交叉引用

Classical and Quantum Nonlocal Gravity

经典与量子非局域引力

- Gauge Invariant Renormalizability of Quantum Gravity

- 量子引力的规范不变可重整性

Nonlocal Gauge Theories Including Quantum Gravity

包含量子引力的非局域规范理论

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